

Charles University in Prague
Faculty of Mathematics and Physics

DIPLOMA THESIS



Jan Kašpar

High energy diffraction processes – TOTEM experiment

Institute of Particle and Nuclear Physics
Supervisor: RNDr. Vojtěch Kunderát, DrSc.
Field of study: Physics, Nuclear and Particle Physics



First of all I would like to thank my supervisors, Drs. Vojtěch Kunderát (for the first part) and Mario Deile (for the second part), for their tireless help and assistance. Next, I would like to thank all my teachers and friends who helped me searching answers to physical questions. Last but not least, a big thank belongs to my family and my girlfriend. They contributed to this thesis by non-physical but essential support.

I declare that I completed my diploma thesis myself and with use of cited literature only. I agree with using this thesis.

Prague, April 15, 2005

Jan Kašpar

Table of contents

Introduction	1
1. Analysis of pp elastic scattering	3
1.1. General concepts	3
1.2. Hadron model	7
1.3. Impact parameter point of view	17
1.4. Coulomb interference	27
1.5. Computation	35
References	36
2. Detector alignment	39
2.1. Testing edgeless strip detectors	39
2.2. Track reconstruction	41
2.3. Calibration	42
2.3.1. Calibration in actual configuration	47
2.3.2. Calibration and detector geometry	48
2.4. Programming	49
References	50

Title: High energy diffraction processes – TOTEM experiment
Author: Jan Kašpar
Department: Institute of particle and nuclear physics
Supervisor: RNDr. Vojtěch Kunderát, DrSc.
Supervisor's
e-mail address: kunderat@fzu.cz

Abstract: We study two problems in this thesis. First, we analyse a model for pp and $\bar{p}p$ elastic scattering. The model was developed by M.M.Islam and coworkers in the past 25 years. Our aim was to make a prediction for differential cross section of pp scattering at energy of 14 TeV which will be measured by the TOTEM experiment at the LHC at CERN. Since protons carry electromagnetic charge, we had to take into account an electromagnetic interaction and effects of the interference between electromagnetic and hadronic forces. We also analysed the model in the impact parameter representation. It enabled us to gain information about range of hadronic forces responsible for elastic, inelastic and total pp and $\bar{p}p$ scattering. In the second part we present our alignment method for detectors inside the Roman pots of the TOTEM experiment. The method was used during Roman pot tests on the SPS beam last year.

Keywords: elastic nucleon scattering, TOTEM experiment

Název práce: Difrakční procesy při vysokých energiích – experiment TOTEM
Autor: Jan Kašpar
Katedra (ústav): Ústav částicové a jaderné fyziky
Vedoucí diplomové
práce: RNDr. Vojtěch Kunderát, DrSc.
e-mail vedoucího: kunderat@fzu.cz

Abstrakt: V této práci jsme se zabývali dvěma problémy. Nejprve jsme analyzovali model pro elastický pp a $\bar{p}p$ rozptyl. Tento model byl vyvinut během minulých 25 let skupinou, kterou vedl M.M.Islam. Naším cílem bylo předpovědět diferenciální účinný průřez pro elastický pp rozptyl při energii 14 TeV, který bude měřen experimentem TOTEM na urychlovači LHC v CERNu. Protože protony nesou elektromagnetický náboj, museli jsme při výpočtech vzít v úvahu elektromagnetickou interakci a efekty způsobené interferencí mezi hadronovou a elektromagnetickou interakcí. Také jsme provedli analýzu zmíněného modelu v prostoru srážkového paramteru. To nám umožnilo získat informaci o dosahu hadronových sil zodpovědných za elastický, neelastický a totální pp a $\bar{p}p$ rozptyl. Ve druhé části jsme se zabývali metodou na kalibraci poloh detektorů, které budou umístěny v římských hrncích experimentu TOTEM. Tato metoda byla použita minulý rok při testech římských hrnců na svazku SPS.

Klíčová slova: elastický rozptyl nukleonů, experiment TOTEM

Introduction

Elastic scattering of hadrons is a process, measurement of which enables us to determine one of the basic characteristics of every hadron collider – the luminosity. Without knowledge of this quantity, it is almost impossible to measure cross sections of other collision processes that are observed. Measurement of the luminosity has always finite precision. And this uncertainty affects precision of the measured cross sections. That is why the aim is to determine luminosity with maximum precision. The rate of elastic scattering is quite high and hence it is possible to gain experimental data with outstanding precision in wide region of momentum transfer.

A new hadron collider, the LHC, is being built at CERN. It will provide colliding protons with center-of-mass energy up to 14 TeV. And it will be the elastic scattering what will be measured first. However, at such a high energy, one can expect that majority of the collisions will be inelastic. But there is a group of inelastic collisions that have similar characteristics to the elastic processes. These processes are called diffraction production. And it is convenient to divide all processes to diffractive and non-diffractive rather than elastic and inelastic ones. All the diffraction processes exhibit a weak energy dependence and the incident particles scatter with small momentum transfer (i.e., they scatter to small angles). Conversely, the non-diffraction processes are strongly energy-dependent and the final state particles have large transverse momenta. The diffractive processes will be studied by the TOTEM experiment. The experiment itself will be able to measure and analyze about 60 % of diffractive channels including the elastic scattering. In cooperation with the CMS detector the ratio of measured diffraction channels can be extended to approximately 95 %.

As already indicated, the diffractive final state contains initial particles beside the newly created particles. And the initial particles are scattered to very small angles. The angles are so small that these particles stay in the accelerator pipe. That is why the TOTEM experiment involves special detector systems called Roman pots. To measure particle tracks in the near forward direction, there will be 3 stations of Roman pots at distances of hundreds of meters from the interaction point. The Roman pot itself is a device that can insert detectors to the pipe and move them close to the beam, when the beam is stabilized. Edge of the detectors will be about 1 mm far from the beam axis. The Roman pots must keep vacuum in the accelerator pipe during their operation.

While some of the inelastic processes are successfully described by perturbation QCD, there is no satisfying theory explaining diffraction phenomena. Hence, one has to use phenomenological models to describe diffraction. These models are built so as to obey rigorous theorems, such as Froissart, Pomeranchuk or Martin theorems. These theorems are usually derived within quantum field theory or analytic S -matrix framework and they usually present a requirement for infinite (asymptotic) energy behavior.

The submitted thesis is devoted to two problems. In the first chapter we analyze a model for pp and $\bar{p}p$ elastic scattering. The model was developed by M.M.Islam and coworkers in the past 25 years. Our aim was to make prediction for differential cross section of pp scattering at energy of 14 TeV which will be measured by the TOTEM experiment. Since protons carry electromagnetic charge, we had to take into account

electromagnetic interaction and effects of interference between electromagnetic and hadron forces. We also analyze the model in the impact parameter representation. It enables us to gain information about the range of hadron forces responsible for elastic, inelastic and total pp and $\bar{p}p$ scattering. In the second chapter we present our alignment method for detectors inside the Roman pots. It was used during Roman pot tests on the SPS beam last year.

In the first chapter we will use natural units, i.e.,

$$\hbar = c = 1 .$$

At the end, we will show how to correct our formulae to satisfy SI unit system.

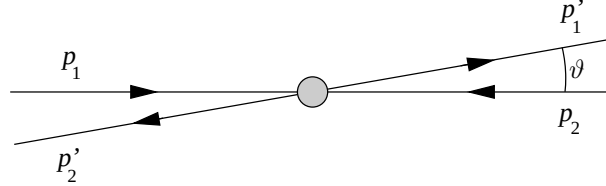
Chapter 1

Analysis of pp elastic scattering

1.1. General concepts

As it was mentioned in the introduction we are interested in pp (or $\bar{p}p$) elastic scattering in this thesis. First of all we have to set a reference frame in which we will investigate these processes. And it is very convenient to chose center-of-mass system (CMS). The situation in this frame is shown in Fig. 1.1.

Fig. 1.1: Kinematics of two body scattering in CMS.



The grey circle represents interaction, p_1 and p_2 are four-momenta of the incoming particles and p_1' and p_2' are four-momenta of the outgoing particles. Due to the mass shell condition, conservation laws and our choice of reference frame, there are actually only 2 independent kinematic parameters (see Ref. [1]). As these two parameters we will use Mandelstam variables s and t

$$s = (p_1 + p_2)^2, \quad t = (p_1' - p_1)^2. \quad (1.1)$$

The variable s denotes the total CMS energy squared and t the four-momentum transfer squared. It is useful to recast definition (1.1) using scattering angle ϑ and momentum p (magnitude of three-momentum) of incident particles in CMS. Assuming both particles have the same mass M , it yields

$$p^2 = \frac{s - 4M^2}{4}, \quad t = -2p^2 (1 - \cos \vartheta). \quad (1.2)$$

With the help of relation (1.2) one finds out there is a restriction for t

$$0 \geq t \geq t_{min} = -4p^2. \quad (1.3)$$

Later, in the section devoted to the impact parameter representation, we will find natural to use quantity

$$q = \sqrt{-t} \quad (1.4)$$

instead of t . This quantity is restricted to region

$$0 \leq q \leq q_{max} = 2p. \quad (1.5)$$

For the square root of s we will use symbol

$$W = \sqrt{s}. \quad (1.6)$$

Having fixed the kinematical description, let us turn to the dynamics of our processes. It is natural to use quantum theory, particularly the interaction picture. Within this framework, the scattering is viewed as a transition from the initial state $|i\rangle$ to the final state $|f\rangle$. Using the S -matrix operator $\hat{S}$ (i.e., the evolution operator in the interaction picture) one can express the relevant amplitude of transition probability as

$$S_{fi} = \langle f | \hat{S} | i \rangle. \quad (1.7)$$

Performing following substitution one can extract all the nontrivial information from S_{fi} and obtain Lorentz invariant amplitude $\mathcal{M}_{fi}$

$$S_{fi} = \langle f | i \rangle + i (2\pi)^4 \delta(p_i - p_f) \mathcal{M}_{fi} \prod_n \frac{1}{(2\pi)^{3/2} \sqrt{2E_n}}, \quad (1.8)$$

where the product over n means multiplication over all particles in the initial and the final state with E_n symbolizing their energy. $\delta(p_i - p_f)$ is four-dimensional delta function expressing four-momentum conservation.

To make a practical use of our calculations it is necessary to enumerate some quantities that are experimentally measurable. Cross section σ is good candidate for this purpose. For two particle scattering in the CMS there is the general formula (see Refs. [2], [3] or [1])

$$d\sigma_{fi} = \frac{1}{4pW} |\mathcal{M}_{fi}|^2 d\Pi_N, \quad (1.9)$$

where $d\Pi_N$ is an element of relativistically invariant phase space of the final state containing N particles

$$d\Pi_N = (2\pi)^4 \delta(p_i - p_f) \prod_{k=1}^N \frac{d^3\mathbf{p}_k}{(2\pi)^3 2E_k}. \quad (1.10)$$

Here $\mathbf{p}_k$ is vector of momentum of the k -th particle in the final state and E_k is the corresponding energy. The δ function governs four-momentum conservation and simultaneously reduces dimension of Π_N to $3N - 4$. Thus the phase space can be parameterized by parameters $a_1, \dots, a_{3N-4}$. Integrating Eq. (1.9) over all of these parameters one obtains the integral cross section $\sigma_{int, fi}$ for interaction $i \rightarrow f$ (here, f denotes particle contents of the final state only)

$$\sigma_{int, fi} = \frac{1}{K} \int da_1 \dots da_{3N-4} \frac{1}{4pW} |\mathcal{M}_{fi}|^2 d\Pi_N(a_j). \quad (1.11)$$

The additional factor K is so called statistical factor and it is equal to $\prod_{\alpha} n_{\alpha}!$ where n_{α} is number of (identical) particles of kind α in the final state.

Summing over all possible final states one gets the total cross section $\sigma_{tot, i}$ for the initial state i

$$\sigma_{tot, i} = \sum_f \sigma_{int, fi} . \quad (1.12)$$

In situations where it will be clear which initial state we have in mind, we will abbreviate $\sigma_{tot, i}$ to $\sigma_{tot}(s)$ (i.e., we will write down the kinematical description of the initial state only).

This general scheme can be applied to the processes we are interested in — the pp and $\bar{p}p$ elastic scattering. As there are two particles in the final state, the phase space Π_N is two-dimensional. Moreover, we are not concerned in spin measurement and that is why we can replace $|\mathcal{M}_{fi}|^2$ in Eq. (1.9) by $\overline{|\mathcal{M}_{fi}|^2}$. It is obtained from $|\mathcal{M}_{fi}|^2$ by averaging over all spin states in the initial state and summing over all spin states in the final state. This spin-averaged squared amplitude is a function of kinematical variables only. And since there is no privileged direction except the scattering axis (axis parallel to $\mathbf{p}_1$ and $\mathbf{p}_2$ in Fig.1.1), $\overline{|\mathcal{M}_{fi}|^2}$ is invariant under rotations around the scattering axis. Therefore the averaged squared amplitude depends on one phase space parameter only. Conveniently, t variable is chosen to play this role and the averaged squared amplitude is written as $\overline{|\mathcal{M}(s, t)|^2}$. In this case, one can perform the integration over the angle round scattering axis in Eq. (1.9) and obtains the formula for differential cross section

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s p^2} \overline{|\mathcal{M}(s, t)|^2} . \quad (1.13)$$

For the elastic interactions with no spin measurement the integral cross section is called elastic cross section and is denoted $\sigma_{el, i}$ (resp. $\sigma_{el}(s)$). It can be evaluated with the help of differential cross section (1.13) as

$$\sigma_{el}(s) = \int_{t_{min}}^0 \frac{d\sigma}{dt} dt . \quad (1.14)$$

The total cross section for collisions of unpolarized particles is given by arithmetic mean of the total cross sections that correspond to particular spin states of colliding particles. Then, one can define the inelastic cross section as

$$\sigma_{inel}(s) = \sigma_{tot}(s) - \sigma_{el}(s) . \quad (1.15)$$

As already said the S matrix operator is evolution operator. That is why it must be unitary. It can be shown that this fact implies (see Ref. [2])

$$\mathcal{M}_{fi} - \mathcal{M}_{if}^* = i(2\pi)^4 \int dn \delta(p_i - p_n) \mathcal{M}_{nf}^* \mathcal{M}_{ni} \prod_{k=1}^{N_n} \frac{1}{(2\pi)^3 2E_k} , \quad (1.16)$$

where $\int dn \delta(p_i - p_n)$ symbolizes integration (and summation) over all states with the same four-momentum as the initial state i , N_n is number of particles in intermediate

state n and E_k is energy of the k -th particle in this state. For $i = f$ Eq. (1.16) leads to the optical theorem which for scattering of unpolarized particles reads

$$\sigma_{tot}(s) = \frac{1}{2pW} \Im \overline{\mathcal{M}}(s, 0), \quad (1.17)$$

where $\overline{\mathcal{M}}(s, 0)$ is elastic amplitude conserving spin state at $t = 0$ GeV² and which is averaged over all initial spin states.

If we have a theory that is PT invariant, i.e., it is invariant under combined space and time inversion, and the initial and the final state contain the same particles with the same spin characteristics, then the amplitude must fulfil $\mathcal{M}_{fi} = \mathcal{M}_{i'f'}$ where the state f' is identical to f but has flipped spins. Thus in theories with no spin or when spin is omitted it must hold

$$\mathcal{M}_{fi} = \mathcal{M}_{if} \quad (1.18)$$

and the l.h.s. of Eq. (1.16) becomes $2\Im \mathcal{M}_{fi}$.

Beside the optical theorem there is number of rigorous theorems that every model (or theory) must obey. We will mention only two of them now. The first one, Froissart theorem (taken from [1]) claims that for asymptotically high energies, i.e., for $s \rightarrow \infty$, the total cross section cannot grow faster than $\ln^2 s$. That is

$$\sigma_{tot}(s) \leq C \ln^2 s, \quad (1.19)$$

where C is a constant.

The second one, Martin's theorem [4] has two assumptions. First, $d\sigma/dt$ for reactions $A + B \rightarrow A + B$ and $A + \bar{B} \rightarrow A + \bar{B}$ must tend to zero for $s \rightarrow \infty$ at t region $-T < t < 0$ for some T . And second, the total cross sections for $A + B$ and $A + \bar{B}$ collisions must tend to infinity for $s \rightarrow \infty$. Then the theorem claims that the real part of crossing even amplitude cannot have constant sign in a strip $S < s$, $-T < t \leq 0$ for any S and $T > 0$. In other words, for every value of s there must exist a value of t such that

$$\Re \mathcal{M}(s, t) = 0 \quad (1.20)$$

if we assume that $\Re \mathcal{M}(s, t)$ is continuous.

Now, we will briefly discuss crossing symmetry. It is based on the fact that (in relativistic QFT) an incoming particle with four-momentum p can be viewed as an outgoing antiparticle with four-momentum $-p$ (see Ref. [1]). This remark can be applied on pp elastic scattering and in terms of invariant amplitudes it reads (see Ref. [5])

$$\mathcal{M}(p(p_1) + p(p_2) \rightarrow p(p_3) + p(p_4)) = \mathcal{M}(p(p_1) + \bar{p}(-p_4) \rightarrow p(p_3) + \bar{p}(-p_2)), \quad (1.21)$$

where the arguments of $\mathcal{M}$ describe the scattering and for example $\bar{p}(-p_4)$ denotes an antiproton with four-momentum $-p_4$. Indeed, if p_4 is momentum of physical particle, $-p_4$ cannot describe a physical particle. Therefore the r.h.s. of Eq. (1.21) is meant in the sense of analytic continuation. Next, one can describe the reaction from l.h.s. of Eq. (1.21) by the Mandelstam variables

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2. \quad (1.22)$$

The same can be done for the r.h.s. reaction (for this reaction we will denote the Mandelstam variables with prime)

$$s' = (p_1 - p_4)^2 = u, \quad t' = (p_1 - p_3)^2 = t, \quad u = (p_1 + p_2)^2 = s. \quad (1.23)$$

Then one can recast Eq. (1.21) to

$$\mathcal{M}_{pp}(s, t) = \mathcal{M}_{\bar{p}p}(u, t). \quad (1.24)$$

It can be shown that $u = 4M^2 - s - t$ and thus for high energies and small values of t one can put

$$u \approx -s, \quad \mathcal{M}_{pp}(s, t) = \mathcal{M}_{\bar{p}p}(-s, t). \quad (1.25)$$

One can define crossing even amplitude $\mathcal{M}^+(s, t)$ and crossing odd amplitude $\mathcal{M}^-(s, t)$

$$\mathcal{M}^\pm(s, t) = \frac{1}{2} \left(\mathcal{M}_{\bar{p}p}(s, t) \pm \mathcal{M}_{pp}(s, t) \right) \quad (1.26)$$

with following crossing property

$$\mathcal{M}^\pm(-s, t) = \pm \mathcal{M}^\pm(s, t). \quad (1.27)$$

The only thing left in order to compute the cross section is to determine the amplitude $\mathcal{M}$. And this is the point where one has to employ models for particular interactions. We are interested in pp and $\bar{p}p$ scattering. For these processes only two of all four known fundamental interactions are relevant. Namely electromagnetic and strong. For computation of electromagnetic amplitude one can use QED, which is very reliable and trusted model. Situation in strong interaction domain is much more complicated. There only exist several phenomenological models and none of them is completely successful. One such a model will be discussed in some detail in the next section.

1.2. Hadron model

We digress a little from the general scheme presented in the previous section and we will introduce an amplitude $T(s, t)$ for elastic unpolarized pp (or $\bar{p}p$) scattering. So, we neglect the spin of nucleons from the very beginning. Moreover, the amplitude $T(s, t)$ will have a different normalization than the invariant amplitude. Then, formulae (1.13) and (1.17) become

$$\frac{d\sigma}{dt} = \frac{\pi}{sp^2} |T(s, t)|^2, \quad (1.28)$$

$$\sigma_{tot}(s) = \frac{4\pi}{pW} \Im T(s, 0). \quad (1.29)$$

Here, in this section, we are going to describe a model of M.M.Islam and coworkers. As there are still new experimental data and new physical information, the model makes progress. Thus there are several stages of this model, but for our purpose 3 of them are relevant. They are represented by papers [6], [7] and [8]. The newer ones

implement more features, but as we will demonstrate, they do not explain older data correctly in some cases.

Although there are several versions of the model, all of them have something in common. It is a notion of the authors how the nucleon structure looks like. It is based on the $SU(3)_L \times SU(3)_R \times U(1)_V$ gauged nonlinear σ model with Wess-Zumino action. Its full treatment can be found in papers [9] and [10]. Here, we want to demonstrate the main result. It can be done with simplified lagrangian density

$$\mathcal{L} = \bar{\psi} i \not{\partial} \psi + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2} \partial_\mu \boldsymbol{\pi} \cdot \partial^\mu \boldsymbol{\pi} - G \bar{\psi} (\sigma + i \boldsymbol{\Lambda} \cdot \boldsymbol{\pi} \gamma^5) \psi - \lambda (\sigma^2 + \boldsymbol{\pi}^2 - f_\pi^2)^2, \quad (1.30)$$

where ψ is a vector of 3 Dirac fields describing quarks, $\boldsymbol{\pi}$ stands for an octet of Goldstone bosons and σ is a scalar boson that plays role of the Higgs particle. $\boldsymbol{\Lambda}$ is a vector of Gell-Mann matrices and f_π is a constant. Then, the derivatives are replaced by covariant derivatives. It is done separately for left and right chiral components of ψ . In the unitary gauge fields $\boldsymbol{\pi}$ disappear and only σ provides interaction between left and right quarks. The authors discovered an interesting critical behavior of σ . If σ is zero at small distances from the origin and sharply rises to its vacuum value f_π at some distance, then the energy of the system of interacting quark Dirac sea with scalar field can be less than that of the noninteracting system. Such a system can make a phase transition to the lower energy state and form the nucleon. In this approach, nucleon is divided into two distinct areas — inner core and outer cloud.

From this result the authors derive how pp (or $\bar{p}p$) collision looks like. They distinguish several mechanisms. Namely, the diffraction, the core scattering and in the latest stages also the quark–quark scattering. For small $|t|$, when only the outer clouds of nucleons overlap, the diffraction dominates. As $|t|$ grows, the inner cores begin to scatter one off the other and it gives rise to the core scattering. For very high $|t|$ the QCD coupling constant is small enough to employ perturbative approach. Then, the qq scattering takes control. Formally, we can describe the situation as

$$T_H(s, t) = T_D(s, t) + T_\omega(s, t) + T_Q(s, t). \quad (1.31)$$

In other words, the complete hadron amplitude $T_H(s, t)$ is given by sum of the diffraction amplitude $T_D(s, t)$, the core scattering amplitude $T_\omega(s, t)$ and the quark–quark scattering amplitude $T_Q(s, t)$.

Next, look at the diffraction a little bit more in detail. The authors parameterize this amplitude in terms of Fourier–Bessel transform

$$T_D(s, t) = ipW \int_0^\infty b db J_0(bq) \Gamma_D^+(s, b), \quad (1.32)$$

where the crossing even profile function is given by

$$\Gamma_D^+(s, b) = g(s) \Gamma_0(s, b) = g(s) \left[\frac{1}{1 + e^{\frac{b-R}{a}}} + \frac{1}{1 + e^{-\frac{b+R}{a}}} - 1 \right]. \quad (1.33)$$

We used abbreviations¹⁾

$$R \equiv R(s) = R_0 + R_1 \left(\ln s - i \frac{\pi}{2} \right), \quad a \equiv a(s) = a_0 + a_1 \left(\ln s - i \frac{\pi}{2} \right) \quad (1.34)$$

with R_0, R_1, a_0, a_1 being free parameters of the model with unit of length. The function $g(s)$ is crossing even, i.e., $g^*(-s) = g(s)$, and asymptotically becomes a real positive constant.

It can be shown that this parameterization satisfies some general theorems. For further discussion we refer the reader to the end of this section.

Let us now turn to the core scattering. Authors claim that the inner cores scatter via ω boson exchange. The ω behaves as an elementary spin-1 boson and thus the amplitude is proportional to $s/(m^2 - t)$, where m is mass of ω . Next, a form factor $F(t)$ and an diffraction absorption factor $1 - \Gamma(s, 0)$ are plugged and the full core scattering amplitude reads

$$T_\omega(s, t) = \pm s \left(1 - \Gamma(s, 0) \right) \hat{\gamma}(s) e^{i\hat{\vartheta}(s)} \frac{F^2(t)}{m^2 - t}. \quad (1.35)$$

The real functions $\hat{\gamma}(s)$ and $\hat{\vartheta}(s)$ express the s dependence of the amplitude and need to be further parameterized. The upper sign holds for $\bar{p}p$ while the lower holds for pp scattering²⁾. For the absorption factor one has to take into account the crossing odd profile function $\Gamma_D^-(s, 0)$

$$\Gamma_D(s, 0) = \Gamma_D^+(s, 0) \pm \Gamma_D^-(s, 0). \quad (1.36)$$

Function $\Gamma_D^-(s, 0)$ is a new free function of the model and need to be parameterized. The form factor $F(t)$ is given by

$$F^2(t) = \beta \sqrt{m^2 - t} K_1 \left(\beta \sqrt{m^2 - t} \right). \quad (1.37)$$

If we stop here, we get to a level described in paper [6]. There are 8 free parameters. a_0, a_1, R_0 and R_1 have dimension of length and are energy-independent. The rest of parameters, $g(s), \hat{\gamma}(s), \hat{\vartheta}(s), \Gamma_D^-(s, 0)$ are dimesionless and they have different values for different values of s . Furthermore, $g(s)$ and $\Gamma_D^-(s, 0)$ are complex quantities and hence there are 10 free real parameters in this stage. Values of β and m are kept fixed at $\beta = 3.075 \text{ GeV}^{-1}$ and $m = 0.801 \text{ GeV}$. To determine values of free parameters, it is necessary to fit the experimental data. This work is done for several energies in paper [6], but without providing method of fitting and values of χ^2 for the fits. That is why we repeated this fitting procedure. But we fitted the data on differential cross section for pp scattering at energy of 53 GeV only (taken from [11]). In this case, when only one process at a single energy is fitted, one can obtain values of 8 parameters only. It is a consequence of the form of $T_D(s, t)$ and $T_\omega(s, t)$. They can be recast to

$$T_D(s, t) = f_D(s) \int_0^\infty b \, db J_0(bq) \Gamma_0(s, b), \quad T_\omega(s, t) = f_\omega(s) \frac{F^2(t)}{m^2 - t}, \quad (1.38)$$

¹⁾ In expressions of type $\ln s$ and s^α , where α is non-integer power, one has to understand s as a dimensionless fraction $s/1 \text{ GeV}^2$.

²⁾ We will keep this sign notation in what follows.

where we factored out only the s -dependent factors

$$f_D(s) = i g(s) p W, \quad f_\omega(s, t) = \pm s \hat{\gamma}(s) e^{i\hat{\vartheta}(s)} \left(1 - \Gamma_D^+(s, 0) \mp \Gamma_D^-(s, 0) \right). \quad (1.39)$$

Now, it is clear that energy-dependent parameters reduce to only two complex free parameters $f_D(s)$ and $f_\omega(s)$ (i.e., to 4 real parameters).

It is important to describe our fitting method. In reality, pp (resp. $\bar{p}p$) scattering is not caused by hadron interaction only. Coulomb interaction is relevant too and as we will show in the last section of this chapter, the Coulomb interaction or Coulomb-hadron interference cannot be neglected for any t value. Therefore we used formula (1.108) to fit experimental data. Comparison of our fit results and the results from [6] is shown in Figs. 1.3 and 1.4 and in Table 1.1 (symbol $\chi^2/D.F.$ means total χ^2 divided by number of degrees of freedom). For completeness we included to the table forward direction (i.e., $t = 0 \text{ GeV}^2$) values of diffraction slope

$$B(s, t) = \frac{d}{dt} \ln |T_H(s, t)|^2. \quad (1.40)$$

and ratio of real and imaginary part of the hadron amplitude

$$\varrho(s, t) = \frac{\Re T(s, t)}{\Im T(s, t)}. \quad (1.41)$$

Table 1.1: Comparison of two fits for pp scattering at energy of 53 GeV.

	R_0 (fm) ³⁾	R_1 (fm)	a_0 (fm)	a_1 (fm)	$\Re f_D$ (fm ⁻²)	$\Im f_D$ (fm ⁻²)
paper [6]	0.311	$1.06 \cdot 10^{-2}$	0.311	$1.06 \cdot 10^{-2}$	$-5.51 \cdot 10^2$	$5.51 \cdot 10^4$
our fit	0.818	$-1.99 \cdot 10^{-2}$	0.721	$-4.79 \cdot 10^{-2}$	$8.22 \cdot 10^3$	$3.05 \cdot 10^4$
	$\Re f_\omega$ (GeV ²)	$\Im f_\omega$ (GeV ²)	$\chi^2/D.F.$	σ_{tot} (mb)	$B(s, 0)$ (GeV ⁻²)	$\varrho(s, 0)$
paper [6]	$-1.50 \cdot 10^2$	$1.48 \cdot 10^3$	102	42.3	17.0	0.0805
our fit	$-1.01 \cdot 10^3$	$-1.11 \cdot 10^3$	1.70	43.0	13.7	0.0734

In Fig. 1.2 we compared experimental data with particular contributions to Islam's amplitude. For this graph we used original amplitude parameters from Ref. [6].

³⁾ We decided to measure length in fm. To convert length from GeV^{-1} (natural units) to fm, one has to multiply the value in natural units by $\hbar c$ (see Eq. (1.120)). This point is discussed a little bit more in detail in the last section devoted to computation.

Fig. 1.2: Differential cross section for pp scattering at energy of 53 GeV with parameters taken from [6]. The blue line represents diffraction alone, the red core scattering alone and the green complete hadron amplitude.

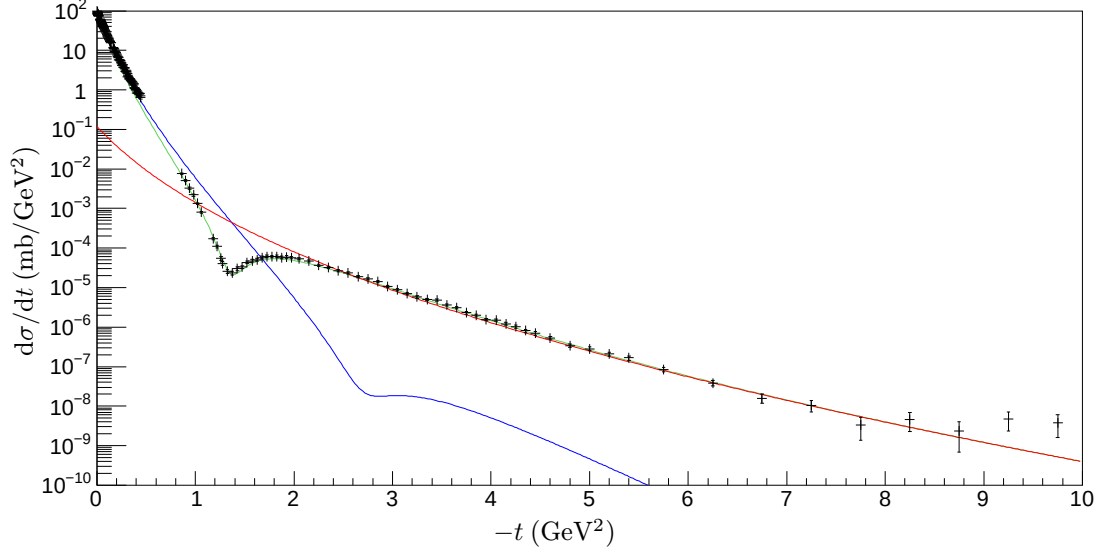


Fig. 1.3: Differential cross section for pp scattering at energy of 53 GeV. The blue curve was plotted with the help of our fit and the red with parameters from [6].

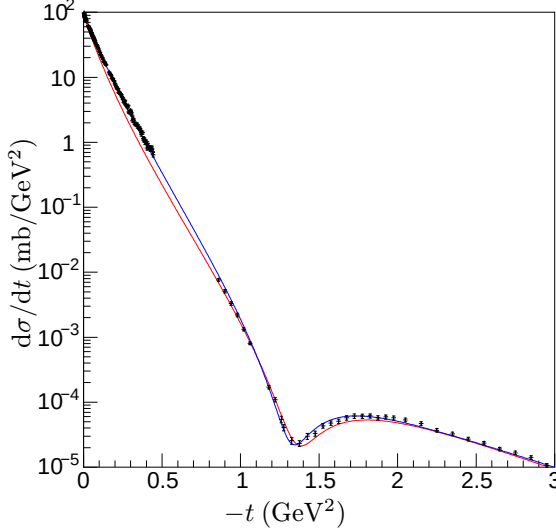
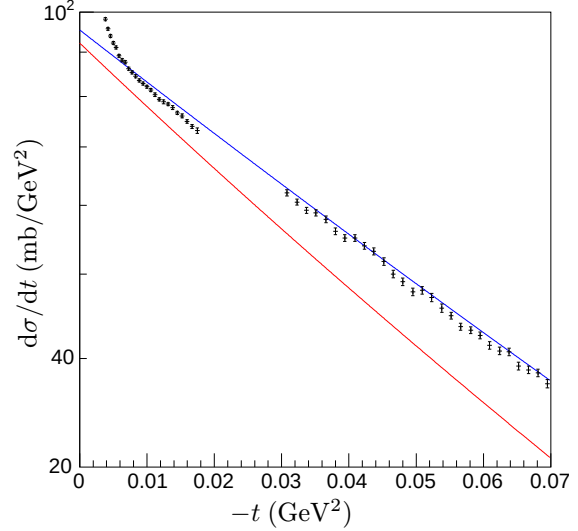


Fig. 1.4: Fig. 1.3 zoomed into the Coulomb region.



In the second stage of the model (paper [7]), the authors parameterize functions $g(s)$, $\hat{\gamma}(s)$, $\hat{\vartheta}(s)$ and $\Gamma_D^-(s, 0)$ by energy-independent parameters. $g(s)$ can be disentangled from relation

$$\eta_0 + \frac{c_0}{(s e^{-i\pi/2})^\sigma} = 1 - \Gamma_D^+(s, 0) \quad (1.42)$$

with the help of definition (1.33) and it yields

$$g(s) = \left(1 - \eta_0 - \frac{c_0}{(s e^{-i\pi/2})^\sigma} \right) \frac{1 + e^{-\frac{R}{a}}}{1 - e^{-\frac{R}{a}}} . \quad (1.43)$$

Crossing odd part of diffraction profile is considered to be

$$\Gamma_D^-(s, 0) = i\lambda_0 - i \frac{d_0}{(s e^{-i\pi/2})^\alpha} . \quad (1.44)$$

$\hat{\gamma}(s)$ and $\hat{\vartheta}(s)$ are parameterized in the following way

$$\hat{\gamma}(s) e^{i\hat{\vartheta}(s)} = \hat{\gamma}_0 + \frac{\hat{\gamma}_1}{(s e^{-i\pi/2})^{\hat{\sigma}}} . \quad (1.45)$$

Putting the previous formulae together, the core scattering amplitude reads

$$T_\omega(s, t) = \pm s \left(\eta_0 + \frac{c_0}{(s e^{-i\pi/2})^\sigma} \mp i\lambda_0 \pm i \frac{d_0}{(s e^{-i\pi/2})^\alpha} \right) \left(\hat{\gamma}_0 + \frac{\hat{\gamma}_1}{(s e^{-i\pi/2})^{\hat{\sigma}}} \right) \frac{F^2(t)}{m^2 - t} , \quad (1.46)$$

In this stage, 9 new free parameters are introduced. Namely, η_0 , c_0 , σ , λ_0 , d_0 , α , $\hat{\gamma}_0$, $\hat{\gamma}_1$ and $\hat{\sigma}$. They are energy-independent and dimensionless. Altogether, with a_0 , a_1 , R_0 and R_1 , the model involves 13 free parameters. To determine their values authors fitted data on $\sigma_{tot}(s)$, $\varrho(s, 0)$ and experimental data on $d\sigma/dt$ for $\bar{p}p$ scattering at energy of 541 GeV. Results can be found in paper [7] and in Fig. 1.5. We repeated the fit for the latter data set, but we used our method based on formula (1.108). The data were taken from [12] (data for energy of 546 GeV), [13] (for 541 GeV) and [14] (for 630 GeV). Comparison of the fits is shown in Table 1.2 and Figs. 1.6 and 1.7.

Table 1.2: Comparison of two fits for $\bar{p}p$ scattering at energy of 541 GeV.

	R_0 (fm)	R_1 (fm)	a_0 (fm)	a_1 (fm)	$\Re f_D$ (fm ⁻²)	$\Im f_D$ (fm ⁻²)
paper [7]	0.493	$7.60 \cdot 10^{-3}$	0.112	$2.09 \cdot 10^{-2}$	$2.69 \cdot 10^5$	$5.62 \cdot 10^6$
our fit	1.325	$-53.4 \cdot 10^{-3}$	-0.683	$8.45 \cdot 10^{-2}$	$-1.36 \cdot 10^5$	$4.85 \cdot 10^6$
	$\Re f_\omega$ (GeV ²)	$\Im f_\omega$ (GeV ²)	$\chi^2/D.F.$	σ_{tot} (mb)	$B(s, 0)$ (GeV ⁻²)	$\varrho(s, 0)$
paper [7]	$-3.23 \cdot 10^4$	$-2.33 \cdot 10^5$	123	62.9	16.8	0.141
our fit	$1.01 \cdot 10^5$	$-2.54 \cdot 10^5$	1.15	63.1	16.1	0.091

Fig. 1.5: Differential cross sections for $\bar{p}p$ scattering at energy of 541 GeV. The blue curve shows diffraction alone, the red core scattering and the green is the complete hadron amplitude. Original parameters from paper [7] were used.

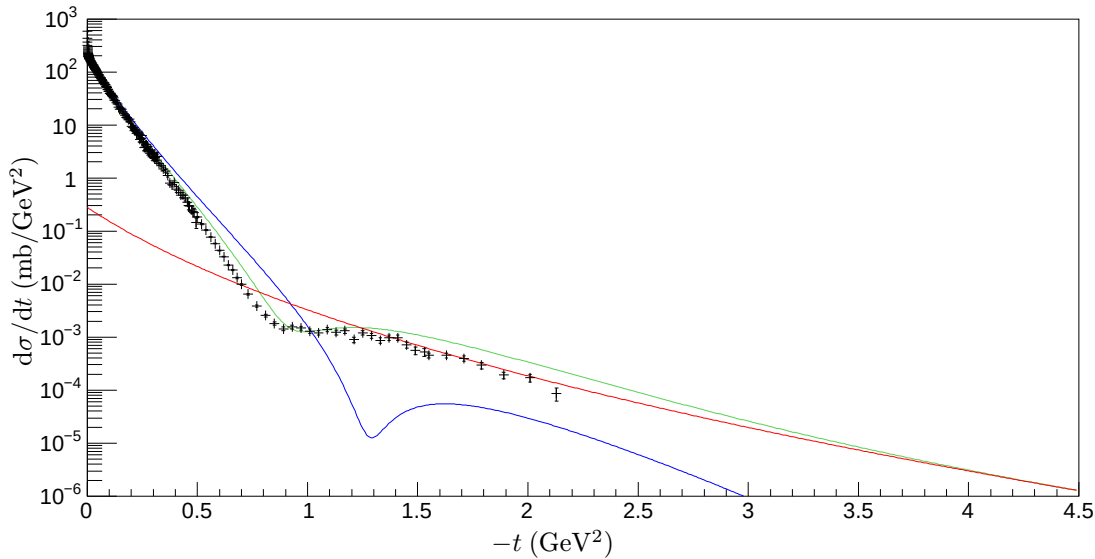


Fig. 1.6: Differential cross sections for $\bar{p}p$ at energy of 541 GeV. Comparison of fit from paper [7] (red line) and our fit represented by Table 1.2 (blue line).

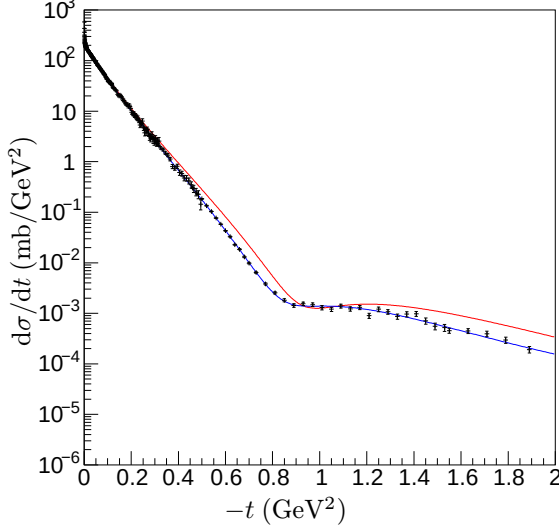


Fig. 1.7: Fig. 1.6 zoomed into Coulomb region.

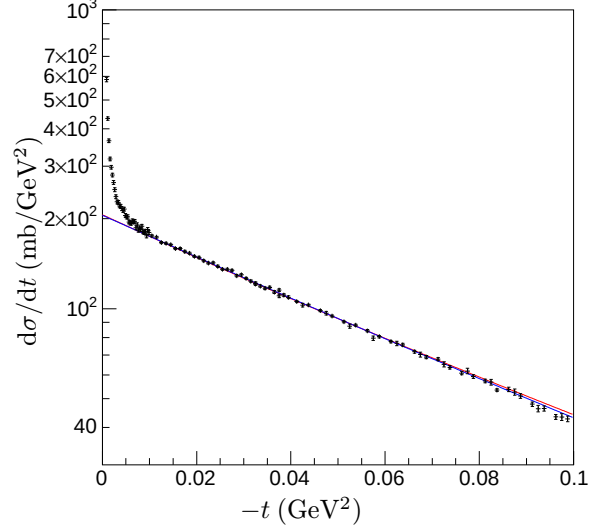


Fig. 1.8: Differential cross sections for $\bar{p}p$ at energy of 541 GeV (blue graph) and for pp at energy of 500 GeV (red graph).

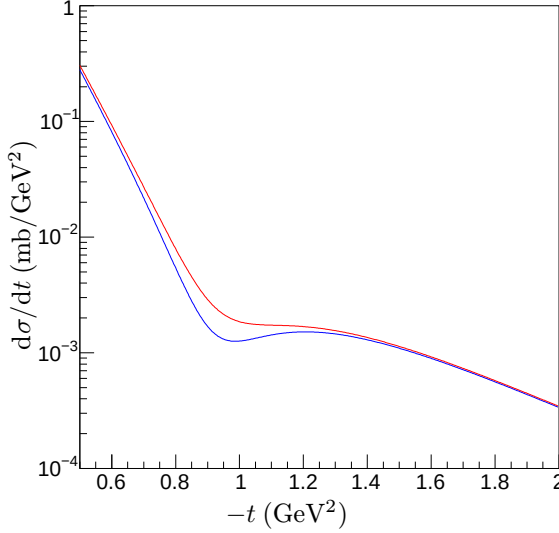
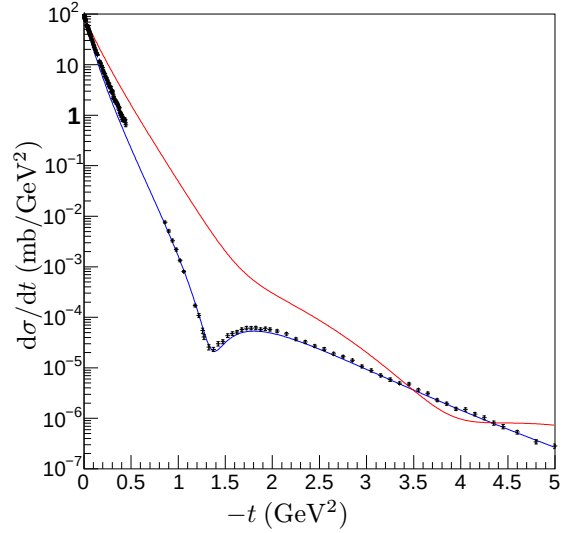


Fig. 1.9: Differential cross sections for pp at energy of 53 GeV. The blue line was obtained with use of stage 1 parameterization, the red line by stage 2 parameterization.



Looking at Figs. 1.5 and 1.6 one can notice that parameter values taken from [7] do not agree with data satisfactorily. Our fit agrees much better (compare values of $\chi^2/D.F.$ in Table 1.2). But note, that we fitted just one data set. Authors of [7] took into account 3 data sets. Fig. 1.8 seems strange a bit since experimental data for pp scattering usually exhibit deeper and narrower dip that data for $\bar{p}p$. Fig. 1.9 shows up that stage 2 parameterization (with parameters from [7]) is unable to explain data on pp scattering at energy of 53 GeV.

Now, let us discuss the extension of the model done in paper [8]. We will refer it as the stage 3. Here, besides diffraction and core scattering a new mechanism is involved. The mechanism is dominant in very high $|t|$ region and reflects transition from the nonperturbative regime to the perturbative regime, where QCD can be used

quite easily. The new mechanism is represented by amplitude $T_Q(s, t)$ in Eq. (1.31).

Within the new mechanism, pp scattering is viewed as a hard collision of valence quarks, one quark from each proton. It brings two new features. First it is a probability amplitude of finding a quark of momentum $\mathbf{P}$ in proton with momentum $\mathbf{p}_1$. We will denote it $\varphi(\mathbf{P})$. And second it is the amplitude for the elastic quark-quark (qq) scattering. To compute this amplitude, the authors did not use directly perturbative QCD but exploited BFKL theory and obtained

$$T_{qq}(s, t) = i \gamma_{qq} s (s e^{-i\pi/2})^\omega \frac{1}{|t| + \frac{1}{r_0^2}}. \quad (1.47)$$

γ_{qq} , ω and r_0 are new free parameters.

Analogically to $\varphi(\mathbf{P})$, one can define $\varphi(\mathbf{K})$ as probability amplitude of a quark to have momentum $\mathbf{K}$ when the proton has momentum $\mathbf{p}_2 = -\mathbf{p}_1$ (see Fig. 1.1). After the collision the quarks have momenta $\mathbf{P} - \mathbf{q}$ and $\mathbf{K} + \mathbf{q}$. They hadronize and create protons with momenta $\mathbf{p}_3 = \mathbf{p}_1 - \mathbf{q}$ and $\mathbf{p}_4 = \mathbf{p}_2 + \mathbf{q}$. Probability amplitudes for such a hadronisation are $\varphi(\mathbf{P} - \mathbf{q})$ and $\varphi(\mathbf{K} + \mathbf{q})$. Note that the momentum transfer is the same on the level of quark and proton scattering. Whereas values of CM energy are different. We will keep the original meaning of s and denote $\tilde{s} = (P + K)^2$. P and K stand for four-momenta corresponding to $\mathbf{P}$ and $\mathbf{K}$. With the help of the probability amplitudes one can write down the qq scattering contribution to the pp scattering amplitude as

$$T'_Q(s, t) = \int d^3\mathbf{P} \int d^3\mathbf{K} \varphi^*(\mathbf{P} - \mathbf{q}) \varphi^*(\mathbf{K} + \mathbf{q}) T_{qq}(\tilde{s}, t) \varphi(\mathbf{P}) \varphi(\mathbf{K}), \quad t = -\mathbf{q}^2. \quad (1.48)$$

Assuming that $\varphi(\mathbf{P})$ is the wave function in momentum representation, one can find the wave function in space representation $\psi(\mathbf{x})$ and obtains

$$\psi(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3\mathbf{P} e^{i\mathbf{P} \cdot \mathbf{x}} \varphi(\mathbf{P}), \quad \int d^3\mathbf{P} \varphi^*(\mathbf{P} - \mathbf{q}) \varphi(\mathbf{P}) = \int d^3\mathbf{x} e^{i\mathbf{q} \cdot \mathbf{x}} \varrho(\mathbf{x}) = F_Q(\mathbf{q}). \quad (1.49)$$

Here we used symbol $\varrho(\mathbf{x}) = |\psi(\mathbf{x})|^2$ for probability density. Then it naturally follows that $F_Q(\mathbf{q})$ can be interpreted as a form factor related to the $\varrho(\mathbf{x})$ density. In [8], a dipole form is taken for this form factor in the proton rest frame

$$F_Q(\mathbf{q}) = \left(1 + \frac{\mathbf{q}^2}{m_0}\right)^{-2} \quad (1.50)$$

with m_0 being the fourth new free parameter in this extension. If one wants to substitute the form factor to Eq. (1.49), one has to use Lorentz contracted form $F_Q(\mathbf{q}_\perp)$. As we are interested in high s limit, we completely neglected the element of $\mathbf{q}$ parallel to $\mathbf{p}$. The part of $\mathbf{q}$ perpendicular to $\mathbf{p}$ is denoted $\mathbf{q}_\perp$.

With the help of Eqs. (1.50) and (1.49) one can extract $\varphi(\mathbf{p})$ and substitute it into Eq. (1.48). Then the integrations can be performed (in high s limit) with the result (rewritten from [8])

$$T'_Q(s, t) = i \tilde{\gamma}_{qq} s (s e^{-i\pi/2})^\omega \frac{\mathcal{F}^2(q_\perp)}{|t| + \frac{1}{r_0^2}}, \quad (1.51)$$

$$\mathcal{F}(q_\perp) = \frac{Mm_0^5}{8\pi} \int_0^1 dx \frac{x^{1+\omega}}{\alpha^2(x)} I(q_\perp, \alpha(x)), \quad \alpha(x) = \sqrt{\frac{m_0^2}{4} + M^2 x^2}, \quad (1.52)$$

$$I(q_\perp, \alpha) = \frac{1}{8\alpha^4} \left(\frac{2}{a^3 a'} \ln(a' + a) + \frac{1}{aa'^3} \ln(a' + a) - \frac{1}{a^2 a'^2} - \frac{3a'}{a^5} \ln(a' + a) + \frac{3}{a^4} \right), \quad (1.53)$$

where

$$a' = \frac{q_\perp}{2\alpha}, \quad a = \sqrt{a'^2 + 1}, \quad q_\perp = \sqrt{t \left(\frac{t}{t_{min}} - 1 \right)}. \quad (1.54)$$

We remind that M denotes proton mass. Note that for $|t| \ll |t_{min}|$ one may approximate $q_\perp \approx \sqrt{-t}$. One more remark. For $q_\perp \rightarrow 0_+$ (i.e., for $t \rightarrow 0_-$) the second and the third term in (1.53) suffer from $q_\perp^{-2}$ divergence. And although they have opposite signs they do not cancel. Finally, the amplitude $T_Q(s, t)$ is given by product of $T'_Q(s, t)$ such that

$$T_\omega(s, t) + T_Q(s, t) = \left(1 - \Gamma_D(s, 0) \right) \left(\hat{\gamma}_0 + \frac{\hat{\gamma}_1}{(s e^{-i\pi/2})^{\hat{\sigma}}} \right) \left[\pm s \frac{F^2(t)}{m^2 - t} + T'_Q(s, t) \right]. \quad (1.55)$$

Alltogether there are 17 free parameters in the model. The new parameters are γ_{qq} , ω (dimensionless), r_0 (dimension of length) and m_0 (dimension of mass). The parameters were determined by fitting the data on $\sigma_{tot}(s)$, $\varrho(s)$ and $d\sigma/dt$ for $\bar{p}p$ at energies $W = 541$ GeV, 630 GeV and 1.8 TeV.

We used this model parameterization and plotted graphs 1.10–1.12. Note that data for $\bar{p}p$ at energy of 541 GeV are much better described than by stage 2 parameterization (Fig. 1.11). However, for pp at energy of 53 GeV stage 3 fails just like stage 2 (Fig. 1.12).

Fig. 1.10: Prediction for differential cross sections for pp scattering at energy of 14 TeV. Diffraction contribution is drawn blue, core scattering red, quark scattering black and complete cross section green.

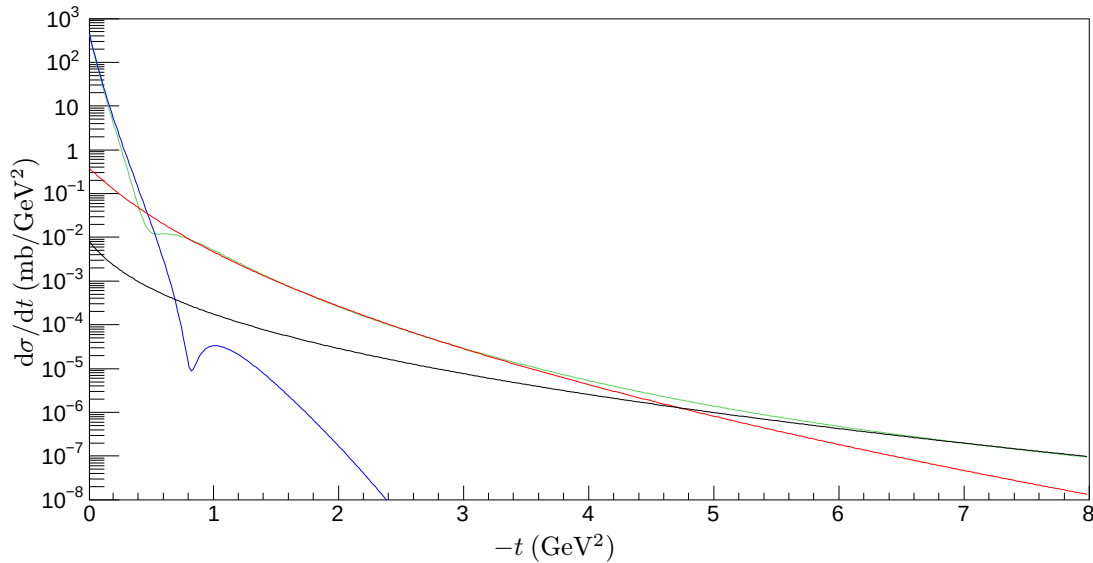


Fig. 1.11: Differential cross sections for $\bar{p}p$ scattering at energy of 541 GeV. The blue line represents stage 2 while the red stage 3.

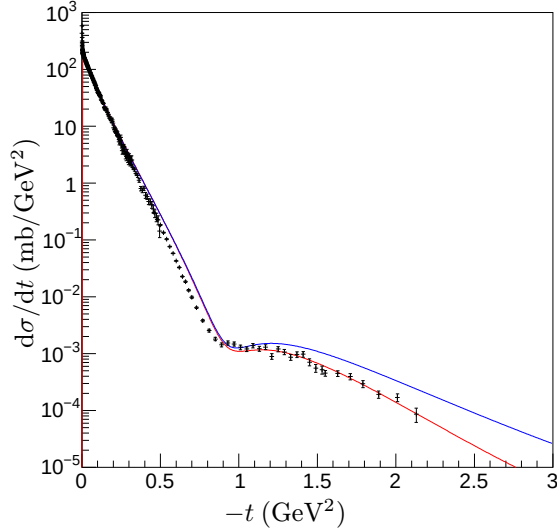
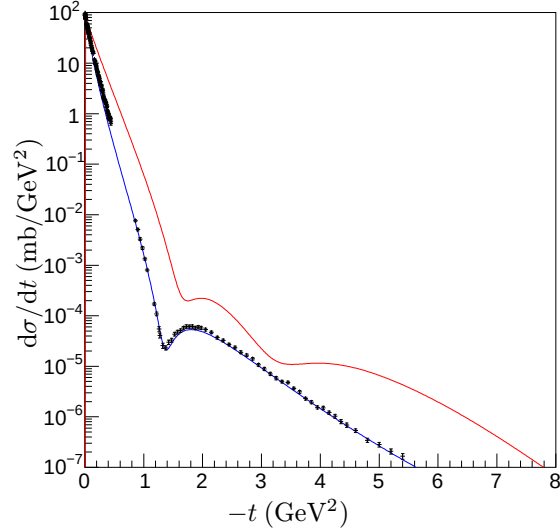


Fig. 1.12: Differential cross sections for pp scattering at energy of 53 GeV. The blue curve is for stage 1 and the red for stage 3.



Now, we will check whether the Islam's model satisfy general theorems. First of all we use optical theorem (1.17) and plot the total cross section, see Fig. 1.13. We do so for all three stages. But since the amplitude $T_Q(s, t)$ diverges at $t = 0$ GeV², we have to exclude it from our considerations. Furthermore, the dominant contribution to $T(s, 0)$ is the diffraction contribution $T_D(s, 0)$. The diffraction amplitude is given by Eqs. (1.32)–(1.34). And it is shown in paper [7] that this parameterization leads to the total cross section proportional to $(a_0 + a_1 \ln s)^2$, what qualitatively saturates the Froissart bound (1.19) at high energies.

Fig. 1.13: Total cross section. The blue line stands for pp and the red for $\bar{p}p$ collision within the stage 2. Stage 3 without $T_Q(s, t)$ is drawn black (pp) and green ($\bar{p}p$).

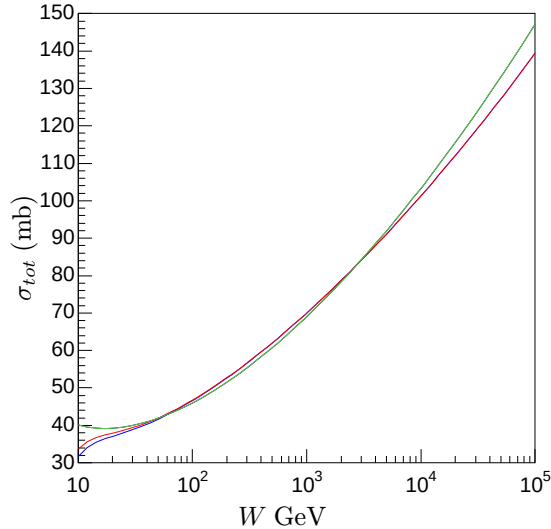
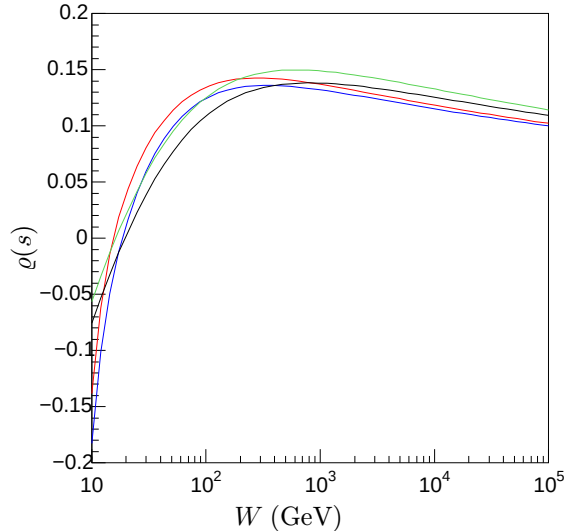


Fig. 1.14: Plot of $\varrho(s)$. The legend is the same as for Fig. 1.13.

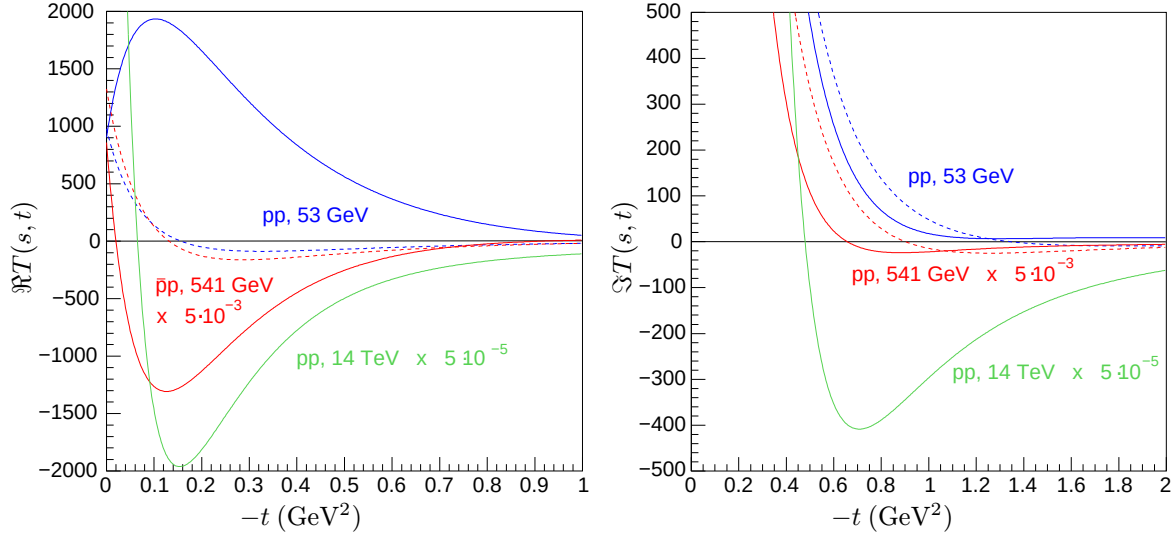


The authors point out that $\varrho(s, 0)$ defined by Eq. (1.41) asymptotically tends to $\pi / \ln s$ as required (see Fig. 1.14). They also demonstrate that assumptions of the Martin theorem (1.20) are fulfilled. Thus there must be a value of t where $\Re T(s, t) = 0$.

It can be checked at Fig. 1.15 that stage 2 and 3 parameterizations have such a zeros at $t \approx -0.1 \text{ GeV}^2$. Stage 1 parameterization crosses zero as well, but at much higher $|t|$.

The zeros of $\Im T(s, t)$ usually correspond to the diffraction dip. It is not true in the case of our fit for $\bar{p}p$ scattering at energy of 541 GeV. Looking at Fig. 1.15 one can see that $\Im T(s, t)$ reaches zero at point $t \approx -0.65 \text{ GeV}^2$. But the diffraction dip takes place at $t \approx -0.9 \text{ GeV}^2$. And this is the point where $\Re T(s, t)$ crosses zero.

Fig. 1.15: The hadron amplitude for 3 sample process (each denoted with different color). The solid lines were obtained with the help of parameters published by Islam et al. The dashed lines correspond to our fits, see Tables 1.1 and 1.2.



1.3. Impact parameter point of view

Working directly with amplitude $T(s, t)$ does not need to be the most efficient way in all situations. And that is why several different representations are employed. In this section we will discuss particularly the impact parameter representation.

From the historical point of view, the impact parameter representation is related with eikonal approximation in nonrelativistic quantum mechanics. In this approach the scattering amplitude $T(s, t)$ is transformed to a different amplitude $A(s, b)$, where the variable b is traditionally called impact parameter. It has a meaning of the transverse distance between colliding particles (i.e., the distance in the plane perpendicular to momenta of incident particles in the CMS). A different way how to derive the impact parameter representation is to make a high energy limit of the partial wave expansion. As it is shown in [1], the impact parameter b is related⁴⁾ to angular momentum value ℓ

$$pb = \ell + \frac{1}{2}. \quad (1.56)$$

And since it holds in the high-energy region, where p and ℓ have large values, the $1/2$ is negligible and one receives classical relation for the impulsmoment ℓ . Thus, the interpretation of b as the impact parameter can be justified.

⁴⁾ Actually, it is defined in this way.

In this section we will follow an exact formulation of impact parameter representation [15] which is valid for all energies and for all values of t . One can check high-energy behavior of the impact parameter amplitude $A(s, b)$ (it is done in Ref. [16]) and finds out that it satisfies

$$A_\ell(s) = A\left(s, b = (2\ell + 1)/2p\right) - \mathcal{O}\left(\frac{1}{p^2}\right), \quad (1.57)$$

where A_ℓ is the partial amplitude corresponding to the angular momentum value ℓ . It is in accordance with Eq. (1.56) and that is why we can keep physical interpretation of b .

The impact parameter representation is based mathematically⁵⁾ on the Fourier-Bessel transformation between the functions $U(q)$ and $A(b)$

$$U(q) = \xi \int_0^\infty b db J_0(bq) A(b), \quad (1.58)$$

$$A(b) = \frac{1}{\xi} \int_0^\infty q dq J_0(bq) U(q), \quad (1.59)$$

where ξ is an arbitrary constant. Now, one can identify function $U(q)$ with amplitude $T(s, -q^2)$ and function $A(b)$ with impact parameter amplitude $A(s, b)$. However, there is a difficulty. Namely, to perform transformation (1.59) one needs to know amplitude $T(s, -q^2)$ in whole integration region. But the physical amplitude is constrained to physical region, that is to $0 < q < q_{max}$. The solution is to define the function $U(s, q)$ as follows

$$U(s, q) = \begin{cases} T(s, -q^2) & \text{for } q < q_{max} \\ \tilde{T}(s, q) & \text{for } q > q_{max} \end{cases}, \quad (1.60)$$

where the function $\tilde{T}(s, q)$ is an arbitrary function, that reflects ambiguity in the impact parameter formulation.

Now we can apply formula (1.59) in a straight-forward way and receive

$$A(s, b) = \frac{1}{\xi} \int_0^{q_{max}} q dq J_0(bq) T(s, -q^2) + \frac{1}{\xi} \int_{q_{max}}^\infty q dq J_0(bq) \tilde{T}(s, q) = a(s, b) + \tilde{a}(s, b), \quad (1.61)$$

where $a(s, b)$ corresponds to the first integral and $\tilde{a}(s, b)$ to the second one. Performing the backward transformation (1.58) one is left with⁶⁾

$$\begin{aligned} U(s, q) &= \xi \int_0^\infty b db J_0(bq) a(s, b) + \xi \int_0^\infty b db J_0(bq) \tilde{a}(s, b) \\ &= T(s, -q^2) \Theta(q_{max} - q) + \tilde{T}(s, q) \Theta(q - q_{max}), \end{aligned} \quad (1.62)$$

⁵⁾ It is worth noticing that the transformation is defined consistently in both directions. It is consequence of the infinite upper bound.

⁶⁾ Symbol $\Theta(x)$ will be hereafter used for a step function.

where again the former term comes from $a(s, b)$ and the latter from $\tilde{a}(s, b)$ transformation. It is clear from this relation that $U(s, q)$ in physical region is unaffected by our choice of $\tilde{T}(s, q)$.

Let us make a short digression from theoretical discussion and look at amplitudes $a(s, b)$ given by Islam model. These amplitudes for two different processes are plotted in Figs. 1.16 and 1.17. Note, that the transformation (1.59) is linear and that is why every contribution to Islam's amplitude $T(s, t)$ (i.e., diffraction, core and quark scattering) has its partner among contributions to $a(s, b)$.

Fig. 1.16: Impact parameter amplitude $a(s, b)$ for pp scattering at energy of 53 GeV (drawn with stage 1 parameterization). The blue curve represents diffraction contribution, the red core scattering contribution and the green complete amplitude $a(s, b)$.

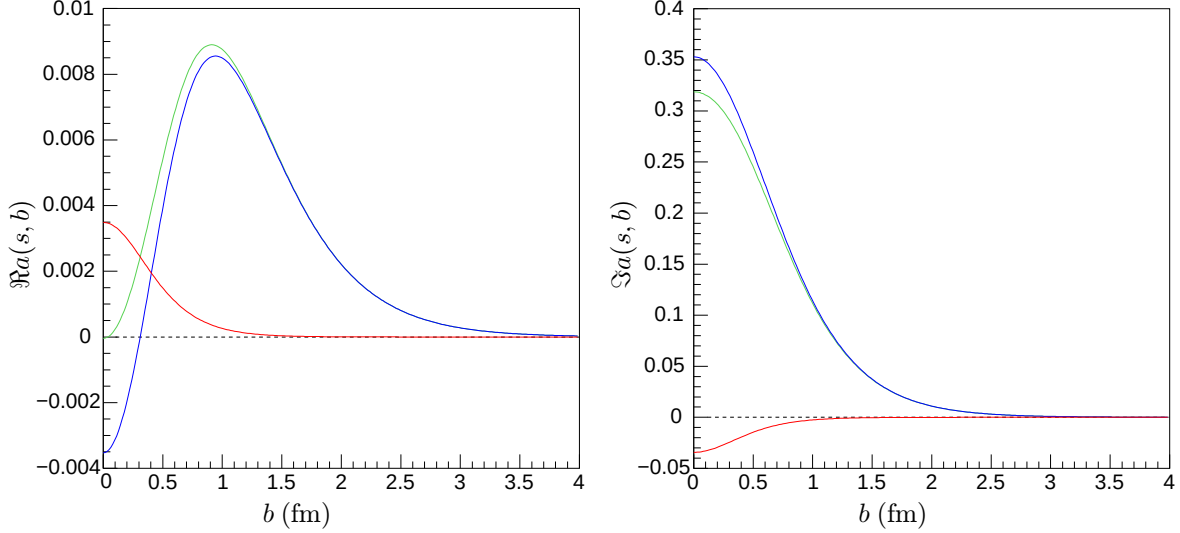
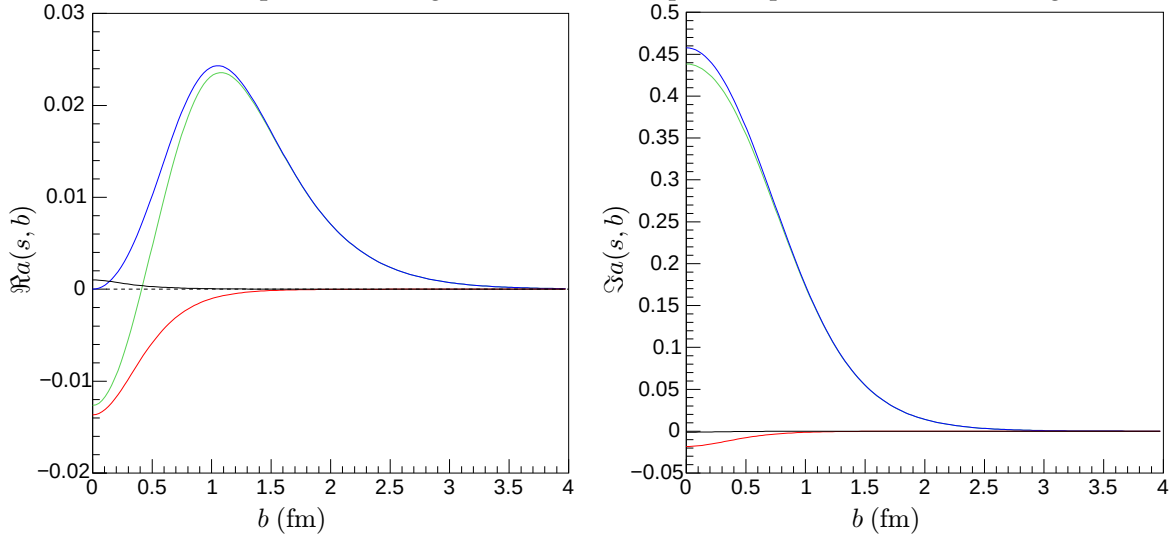


Fig. 1.17: Impact parameter amplitude $a(s, b)$ for $\bar{p}p$ scattering at energy of 541 GeV (drawn with stage 3 parameterization). The blue solid line denotes diffraction contribution, the red core and the black solid line quark scattering contribution. Complete amplitude $a(s, b)$ is drawn green.



It is a typical feature of the Islam model that the amplitude $T(s, t)$ is almost imaginary for $|t|$ below the diffraction dip. In this region we can put $\Re T(s, t) \ll \Im T(s, t) \approx |T(s, t)|$ and deduce behavior of the amplitude from experimental data

which can be approximately described by $|T(s, t)| \sim \exp(-Bt)$ in that region (B is appropriate constant). Now, it is clear that decisive contribution to $\Im a(s, b)$ (see Eq. (1.61)) comes from the mentioned small t region where diffraction is in charge. It explains why the diffraction contribution dominates in the right graphs of Figs. 1.16 and 1.17. The approximately gaussian shape can be explained as well. The transformation (1.61) of $\Im T(s, t) \sim \exp(-Bt)$ can be carried out for $s \rightarrow \infty$ and one obtains $\Im a(s, b) \sim \exp(-Cb^2)$, where C is a constant.

As it was said above, the impact parameter amplitude is ambiguous. To remove this ambiguity one can introduce some requirements that determine the amplitude $A(s, b)$ fully. Before doing this we will derive some relations that will help us to find both, the requirements and a physical interpretation of amplitude $A(s, b)$. Let us start with formula for the total cross section $\sigma_{tot}(s)$. Substituting $U(s, q)$ from Eq. (1.62) for $T(s, -q^2)$ into optical theorem (1.29) one gets

$$\sigma_{tot}(s) = \frac{4\pi\xi}{pW} \int_0^\infty b db \Im A(s, b) = \frac{4\pi\xi}{pW} \int_0^\infty b db \Im a(s, b) , \quad (1.63)$$

where the first equality is a consequence of $J_0(0) = 1$. The second equality follows from the fact that in optical theorem we take the value of the amplitude in forward direction ($t = 0 \text{ GeV}^2$). That is at (the boundary of) physical region and thus only the first term in (1.62) contributes. In other words $\tilde{a}(s, b)$ has following property

$$\int_0^\infty b db \Im \tilde{a}(s, b) = 0 . \quad (1.64)$$

One can derive a formula similar to Eq. (1.63) for the elastic cross section $\sigma_{el}(s)$ as well. A feasible way is to begin with definition of the elastic cross sections (1.14) and (1.28). The square of amplitude in the latter equation can be recast to $T^*(s, t) T(s, t)$ and the factor $T^*(s, t)$ can be substituted from Eq. (1.62). Note that Eq. (1.14) contains integral through physical region only. And that is why it is sufficient to take only the first term from decomposition (1.62). However, considering the full amplitude $U(s, q)$ is relevant as well. Thus, one can perform the suggested substitution once with $U(s, q)$ and with $T(q) \Theta(q_{max} - q)$ for the second time. It leads to following chain of equalities

$$\sigma_{el}(s) = \frac{2\pi\xi^2}{sp^2} \int_0^\infty b db |a(s, b)|^2 = \frac{2\pi\xi^2}{sp^2} \int_0^\infty b db A^*(s, b) a(s, b) , \quad (1.65)$$

that hides a constraint

$$\int_0^\infty b db \tilde{a}^*(s, b) a(s, b) = 0 \quad (1.66)$$

and thus $a(s, b)$ and $\tilde{a}(s, b)$ are not independent.

Looking at formulae (1.63) and (1.65) one can find, that fixing $\xi = 2pW$ is very convenient. Then that formulae unify to

$$\sigma_{tot}(s) = 8\pi \int_0^\infty b db \Im A(s, b) , \quad \sigma_{el}(s) = 8\pi \int_0^\infty b db |a(s, b)|^2 . \quad (1.67)$$

In fact there is one more advantage of this choice. Particularly, the impact parameter amplitude is dimensionless, just like the common amplitude $T(s, t)$. That is why we will use this convention.

Graph of $\sigma_{tot}(s)$ for Islam's model is plotted in Fig. 1.13 in the previous section. Graph of $\sigma_{el}(s)$ is shown in Fig. 1.21 a few pages later.

At this moment we are ready to discuss the meaning of impact parameter amplitude. For this purpose let us introduce a vector $\mathbf{b}$ denoting relative position of the incident particles in the transversal plane. That is plane perpendicular to momenta of colliding particles in the CMS. Then the impact parameter b is magnitude of the vector $\mathbf{b}$. We can rewrite relations (1.67) employing integration over whole transversal plane

$$\sigma_{tot}(s) = \int d^2\mathbf{b} 4\Im A(s, b), \quad \sigma_{el}(s) = \int d^2\mathbf{b} 4|a(s, b)|^2. \quad (1.68)$$

This result evokes a clear interpretation. The total resp. the elastic cross section can be obtained by integrating corresponding density over the whole transversal plane (i.e., over all impact parameter configurations). The densities ϱ_{tot} and ϱ_{el} for the total and the elastic cross sections are

$$\varrho_{tot}(s, b) = 4\Im A(s, b), \quad \varrho_{el}(s, b) = 4|a(s, b)|^2. \quad (1.69)$$

However, if we want to interpret these functions as densities of collisions (elastic or total⁷⁾), these functions must be non-negative. While this condition is fulfilled automatically in the case of the elastic density $\varrho_{el}(s, b)$, for the total density $\varrho_{tot}(s, b)$ it is not generally satisfied. Of course, when we require $\varrho_{tot}(s, b)$ to be non-negative, it leads to restrictions for $\tilde{a}(s, b)$ resp. $\tilde{T}(s, q)$. And it has been our goal.

One may question if there generally exists a function $\tilde{T}(s, q)$ that satisfies

$$\varrho_{tot}(s, b) \geq 0. \quad (1.70)$$

The second question is whether this function is unique. And the final question is how to find such a function. Similar problem was studied in paper [17]. They focused on the last question and successfully applied a numerical approach. They directly searched for $\Im\tilde{a}(s, b)$ ($c(s, b)$ in their notation) instead of $\tilde{T}(s, q)$. It is easy to understand since $\Im\tilde{a}(s, b)$ performs directly in $\varrho_{tot}(s)$ definition. On the other hand it is necessary to fulfill properties of $\Im\tilde{a}(s, b)$ such as (1.64) and (1.66). In the quoted paper only Eq. (1.64) is kept in mind. And as we will demonstrate in a sequel, there is infinite number of constraints for $\tilde{a}(s, b)$. Thus the approach based on finding $\Im\tilde{a}(s, b)$ gets into troubles.

Hence finding appropriate function $\tilde{T}(s, q)$ is a difficult task. One may wonder if there are some quantities that would give us some information about densities $\varrho_{tot, el}(s, b)$ and simultaneously we would not need to determine $\tilde{T}(s, q)$ for their evaluation. The answer is there are such quantities. As an example we mention mean values of b^2 for elastic or total collisions

$$\langle b^2(s) \rangle_{el, tot} = \frac{\int d^2\mathbf{b} b^2 \varrho_{el, tot}(s, b)}{\int d^2\mathbf{b} \varrho_{el, tot}(s, b)} = \frac{1}{\sigma_{tot, el}(s)} \int d^2\mathbf{b} b^2 \varrho_{el, tot}(s, b). \quad (1.71)$$

⁷⁾ The total collision means a collision of whatever type, i.e., elastic or inelastic.

Now we will demonstrate how to calculate their values. To do so, we will make use of following identities for Bessel functions with integer m (taken from [18])

$$\frac{d}{dx}(x^m J_m(x)) = x^m J_{m-1}(x), \quad J_{-m}(x) = (-1)^m J_m(x), \quad (1.72)$$

$$\lim_{x \rightarrow 0} \frac{J_m(x)}{x^m} = \frac{1}{2^m m!} \quad \text{for } m \geq 0. \quad (1.73)$$

Equipped with these relations it is easy to check (for integer $n \geq 0$)

$$\lim_{q \rightarrow 0_+} \left(\frac{1}{q} \frac{d}{dq} \right)^n U(s, q) = \frac{(-1)^n}{2^n n!} \int_0^\infty b db b^{2n} A(s, b). \quad (1.74)$$

In this equation both terms on r.h.s. of (1.62) are taken into account. But since we take (one side) limit at point $t = 0 \text{ GeV}^2$ (and limit and derivative are local operations) only the first term, containing $a(s, b)$ contributes. That is why one can replace $A(s, b)$ by $a(s, b)$ in Eq. (1.74). It means

$$\int_0^\infty b db b^{2n} \tilde{a}(s, b) = 0. \quad (1.75)$$

Comparing Eqs. (1.74) and (1.71) for the total case, one can notice that numerator of Eq. (1.71) is proportional to imaginary part of (1.74) with $n = 1$ and denominator to the same expression with $n = 0$. Writing down this observation one can obtain compact form (copied from [17])

$$\langle b^2(s) \rangle_{tot} = 4 \frac{d}{dt} \ln \Im T(s, t) \Big|_{0_-}. \quad (1.76)$$

This mean value for model of Islam is plotted in Fig. 1.18. In paper [17] the authors derived an approximate relation for $\langle b^2(s) \rangle_{tot}$

$$\langle b^2(s) \rangle_{tot} \approx 4 \frac{d}{dt} \ln |T(s, t)| \Big|_{0_-}. \quad (1.77)$$

The advantage of this form is that it could be evaluated directly from experimental data. Comparison of this formula and exact formula (1.76) is shown in Fig. 1.20.

Eq. (1.75) enables us to make an important conclusion about $\langle b^2(s) \rangle_{tot}$. Namely, it is independent on choice of $\tilde{T}(s, q)$. Hence, the existence of function $\tilde{T}(s, q)$ satisfying (1.70) is necessary only for correct interpretation of $\langle b^2(s) \rangle_{tot}$ as a mean value of b^2 .

Let us turn to the elastic mean b^2 now. Here, we will confine to physical region from the very beginning. Then we can compute derivative⁸⁾

$$\frac{dU(s, q)}{dq} = \frac{dT(s, -q^2)}{dq} = - \int_0^\infty b^2 db J_1(bq) a(s, b) \quad 0 \leq q < q_{max}. \quad (1.78)$$

⁸⁾ Note that derivative of ϑ functions in $U(s, q)$ definition (1.62) would give rise to corresponding δ functions. The restriction to physical region enables us to forget the δ functions.

With the help of this result one gets readily

$$\int_0^{q_{max}} q dq \left| \frac{dT(s, -q^2)}{dq} \right|^2 = \int_0^\infty b db b^2 |a(s, b)|^2. \quad (1.79)$$

The r.h.s. of just received equation is proportional to (1.71) for the elastic collisions. Thus (1.71) can be rewritten with the use of the last equation and t variable (the form was taken from Ref. [19])

$$\langle b^2(s) \rangle_{el} = 4 \frac{\int_{t_{min}}^0 dt |t| \left| \frac{d}{dt} T(s, t) \right|^2}{\int_{t_{min}}^0 dt |T(s, t)|^2}. \quad (1.80)$$

Fig. 1.18: Total RMS⁹⁾ of b . The blue line corresponds to pp and the red to $\bar{p}p$ scattering with parameterization of stage 2 of Islam's model. Similarly, the black line stands for pp and the green for $\bar{p}p$ scattering with stage 3 parameterization.

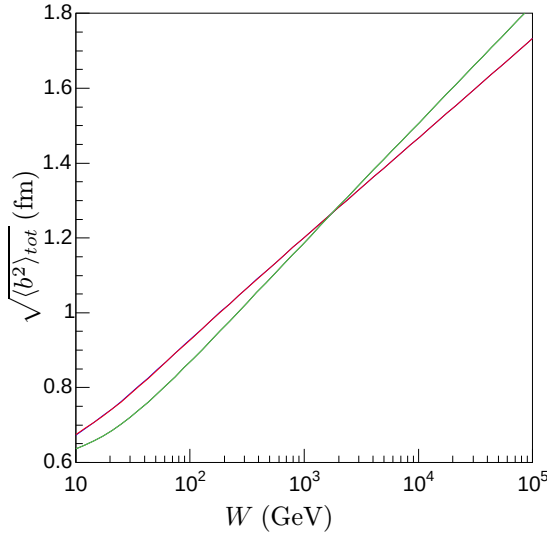
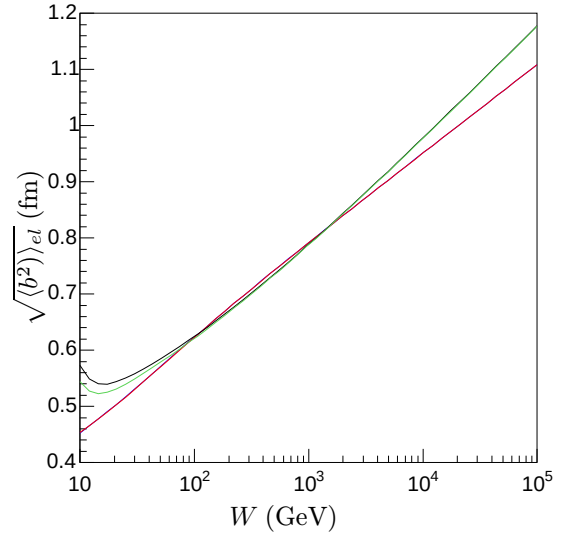


Fig. 1.19: RMS of b for the elastic collisions. Legend is the same as in Fig. 1.18.



⁹⁾ RMS is abbreviation of root mean square. That is, the total RMS of b means $\sqrt{\langle b^2 \rangle_{tot}}$.

Fig. 1.20: Total RMS of b . The black curves are obtained by exact formula (1.76) while the green by approximate formula (1.77). The solid curves use stage 2 and the dashed stage 3 parameterization.

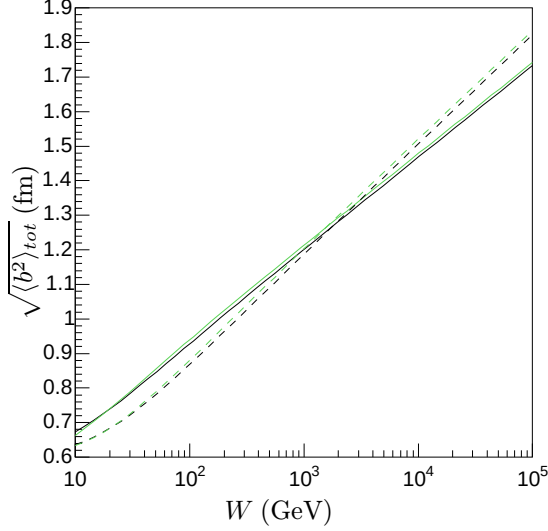
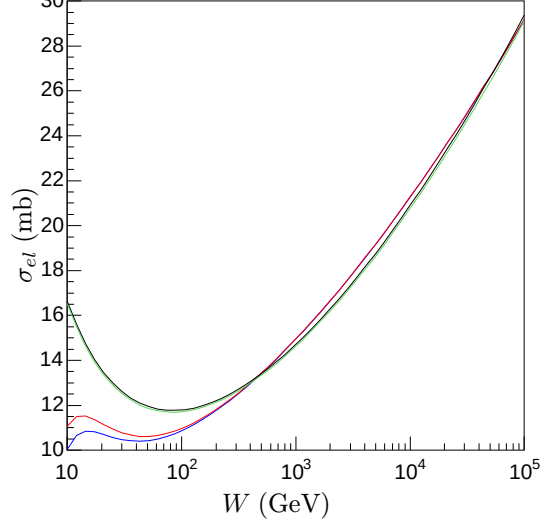


Fig. 1.21: Elastic cross section. See legend of Fig. 1.18.



The fact, that mean b^2 for the total collisions is greater than for the elastic collisions (compare Figs. 1.18 and 1.19) is typical for so called central models which the Islam model belongs to. These models have a weak t dependence of the phase of the amplitude in the small $|t|$ region. Whereas models with a convenient strong t dependence of phase lead to so called peripherality and yield mean b^2 greater for the elastic than for the total collisions (for details see Ref. [20]).

At the end of this section we would like to devote some space to the unitarity relation in the impact parameter representation. We start from Eq. (1.16) (adapted for the elastic scattering) and adopt assumption (1.18). We recast Eq. (1.16) to form

$$\Im T(s, t) = E(s, t) + F(s, t) , \quad (1.81)$$

where the $E(s, t)$ term contains all elastic intermediate states n and the $F(s, t)$ involves the rest. Indeed, the $E(s, t)$ term is expressible as function of the elastic amplitude $T(s, t)$. Eq. (4.5) from [15] reads

$$E(s, t) = \frac{1}{8pW} \int_0^\infty b db J_0(b\sqrt{-t}) \int_{t_{min}}^0 dt_1 \int_{t_{min}}^0 dt_2 T^*(s, t_1) T(s, t_2) L(b; t_1, t_2) , \quad (1.82)$$

$$L(b; t_1, t_2) = J_0\left(b\sqrt{t_1\left(\frac{t_2}{t_{min}} - 1\right)}\right) J_0\left(b\sqrt{t_2\left(\frac{t_1}{t_{min}} - 1\right)}\right) . \quad (1.83)$$

For the elastic scattering the interesting region of momentum transfer t (the experimentally measured region) is $|t| \ll |t_{min}|$. In this region we can approximate

$$L(b; t_1, t_2) \approx J_0(b\sqrt{-t_1}) J_0(b\sqrt{-t_2}) \quad (1.84)$$

and for fixed t_1 and t_2 the approximation is the better the higher $|t_{min}|$ (i.e., the higher s) we have. Later we will find convenient to define a correction function $M(b; t_1, t_2)$

$$L(b; t_1, t_2) = J_0(b\sqrt{-t_1}) J_0(b\sqrt{-t_2}) + M(b; t_1, t_2) , \quad M(b; t_1, t_2) \approx 0 . \quad (1.85)$$

Now, we would like to apply transformation (1.59) on unitarity relation (1.81). But since functions $E(s, t)$ and $F(s, t)$ are defined at physical region only we get into troubles. One can get over this problem in a similar way as with amplitude $T(s, t)$ and in analogy with (1.60) introduce arbitrary functions $\tilde{E}(q)$ and $\tilde{F}(q)$. Following the analogy (see Eq. (1.61)) we will denote transformation of $E(s, t)$ resp. $F(s, t)$ by $e(s, b)$ and $f(s, b)$. Non-physical contributions will be denoted $\tilde{e}(b)$ and $\tilde{f}(b)$. Performing the transformation of (1.81) separately in physical and non-physical region one obtains set of Eqs.

$$\Im a(s, b) = e(s, b) + f(s, b) , \quad (1.86)$$

$$\Im \tilde{a}(s, b) = \tilde{e}(b) + \tilde{f}(b) . \quad (1.87)$$

As the function $\tilde{E}(t)$ is arbitrary we can put

$$\tilde{E}(t) \equiv 0 \quad (1.88)$$

and obtain

$$\Im A(s, b) = e(s, b) + G(s, b) , \quad (1.89)$$

where we used $G(s, b) = f(s, b) + \tilde{f}(b)$.

The function $e(s, b)$ can be calculated directly from Eqs. (1.82) and (1.85) with result

$$e(s, b) = |a(s, b)|^2 + K(s, b) , \quad (1.90)$$

where $|a(s, b)|^2$ comes from the first term in (1.85) and the correction function $K(s, b)$ comes from the second term. It is explicitly given as

$$K(s, b) = \frac{1}{16sp^2} \int_{t_{min}}^0 dt_1 \int_{t_{min}}^0 dt_2 T^*(s, t_1) T(s, t_2) M(b; t_1, t_2) . \quad (1.91)$$

If one takes complex conjugate of $K(s, b)$ and subsequently swaps the integration variables in (1.91), one obtains $K^*(s, b) = K(s, b)$ and hence the correction function is real.

Putting (1.89) and (1.90) together it yields the desired unitarity relation in the impact parameter space

$$\Im A(s, b) = |a(s, b)|^2 + G(s, b) + K(s, b) , \quad (1.92)$$

where the term $K(s, b)$ is negligible in high s limit. We plotted the correction function given by Islam's model for two processes in Fig. 1.22. And in comparison with imaginary part of $a(s, b)$ (Figs. 1.16 and 1.17) we see, it is suppressed by factor grater than 10^5 . Fig. 1.22 also confirms the assumption that for higher s the contribution of $K(s, b)$

should be smaller. In later applications of the unitarity relation we will neglect the correction $K(s, b)$.

One may multiply Eq. (1.92) by b^3 and integrate over b from 0 to ∞ and receive

$$\sigma_{tot}(s) \langle b^2(s) \rangle_{tot} = \sigma_{el}(s) \langle b^2(s) \rangle_{el} + \sigma_{inel}(s) \langle b^2(s) \rangle_{inel} \quad (1.93)$$

where we denoted (in accordance with Eqs. (1.68) and (1.71))

$$\sigma_{inel}(s) = \int d^2\mathbf{b} 4G(s, b) \quad \langle b^2(s) \rangle_{inel} = \frac{\int_0^\infty b db b^2 4G(s, b)}{\int_0^\infty b db 4G(s, b)} . \quad (1.94)$$

With the help of Eqs. (1.68) and (1.92) one can check that the previous formula agrees with the original definition of $\sigma_{inel}(s)$ (1.15). $\langle b^2(s) \rangle_{inel}$ is defined in analogy with (1.71) and we would like to interpret it as a mean value of b^2 for the inelastic collisions. And again, we face the same interpretation problem as in the case of the total mean b^2 . The corresponding collision density $\varrho_{inel}(s, b) = 4G(s, b)$ need not be positive. The solution might be following. Since we adopted (1.88), $\tilde{f}(b) = \Im \tilde{a}(s, b)$. And there might exist such a function $\tilde{T}(s, q)$ that would guarantee both, condition (1.70) and

$$\varrho_{inel}(s, b) \geq 0 . \quad (1.95)$$

In fact, $\langle b^2(s) \rangle_{inel}$ is independent of choice of $\tilde{F}(q)$ (just like $\langle b^2(s) \rangle_{tot}$ is independent on $\tilde{T}(s, q)$) and condition (1.95) is important only for the suggested interpretation. However, if we adopt (1.95), all terms in (1.93) become non negative and one can gain inequalities¹⁰⁾

$$\langle b^2(s) \rangle_{el} \leq \frac{\sigma_{tot}}{\sigma_{el}} \langle b^2 \rangle_{tot} = \langle b^2(s) \rangle_{el. bound} \quad (1.96)$$

$$\langle b^2(s) \rangle_{inel} \leq \frac{\sigma_{tot}}{\sigma_{inel}} \langle b^2 \rangle_{tot} = \langle b^2(s) \rangle_{inel. bound} \quad (1.97)$$

The elastic bound is compared to $\langle b^2(s) \rangle_{el}$ (for the Islam's model) in Fig. 1.23.

¹⁰⁾ These inequalities can be obtained without choice (1.88).

Fig. 1.22: Modulus of the correction function $K(s, b)$ for two sample processes.

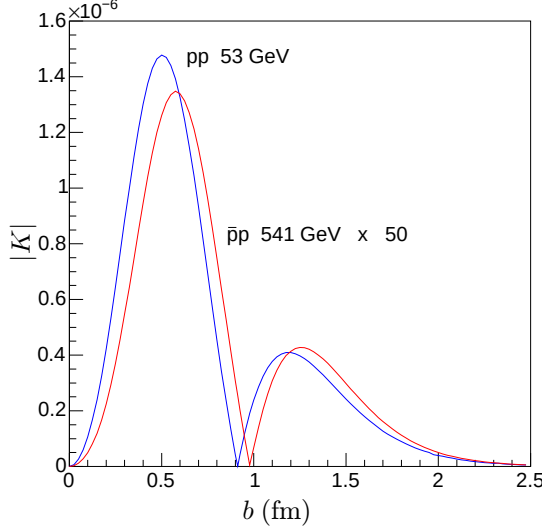
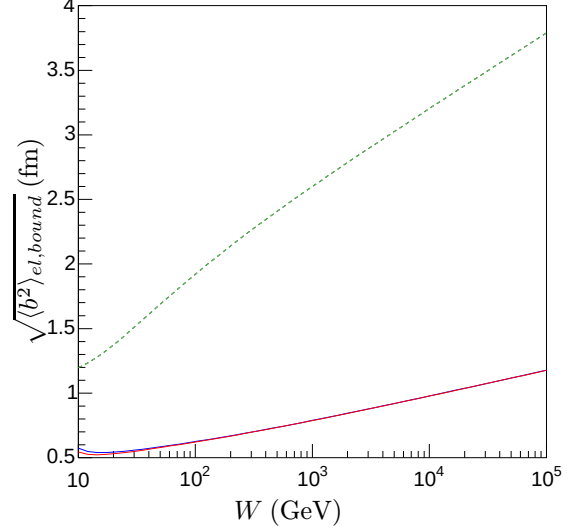


Fig. 1.23: Bound (1.96) for the elastic RMS of b (the black line represents pp scattering parameterized by stage 1 and the green by stage 3). It is compared with the elastic RMS of b with use of stage 3. The blue line is for pp and the red for $\bar{p}p$ scattering.



A conclusion of this section. We presented an impact parameter formalism where the impact parameter amplitude is ambiguous. The ambiguity is expressed by functions $\tilde{T}(s, q)$, $\tilde{E}(q)$ and $\tilde{F}(q)$ defined in the non-physical region. We raised natural requirements (1.70), (1.95), (1.88) on this functions. The last requirement can be generally satisfied. Questions, whether the first two requirements can be fulfilled and whether they determine function $\tilde{T}(s, q)$ uniquely, are left open in this thesis. For their treatment we refer the reader to [21].

1.4. Coulomb interference

So far we have omitted an influence of electromagnetic interaction in pp resp. $\bar{p}p$ scattering. We will remedy it in this section. Generally, one can expect two effects at the amplitude level. First, it is an appearance of amplitude describing pure electromagnetic scattering and second, it is a contribution arising from electromagnetic and hadron interaction acting simultaneously. Denoting the former $T_C(s, t)$ ¹¹⁾ and the latter $R(s, t)$, the full scattering amplitude $T(s, t)$ can be written

$$T(s, t) = T_C(s, t) + T_H(s, t) + R(s, t), \quad (1.98)$$

where $T_H(s, t)$ is amplitude for pure hadron scattering discussed in preceding sections. Basically, there are two approaches for calculation of the electromagnetic effects. The first is based on quantum mechanics in eikonal approximation and the second on evaluating some relevant Feynman graphs.

¹¹⁾ As it is common, we will use Coulomb amplitude as synonym to electromagnetic amplitude and so on.

Let us begin with the quantum mechanical approach. Here, the scattering is viewed as a scattering of one particle off potential of the second particle. So it is different from the picture described in the beginning of this chapter. However, it can be accommodated. Here, we will consider only spherically symmetric potentials $V(r)$. Furthermore, for high energy scattering, one can employ the eikonal model¹²⁾. It gives the following prescription for scattering amplitude¹³⁾

$$T(s, t) = 2pW \int_0^\infty b db J_0(b\sqrt{-t}) \frac{e^{2i\delta(s, b)} - 1}{2i}, \quad \delta(s, b) = -\frac{1}{2p} \int_0^\infty dz V(\sqrt{b^2 + z^2}), \quad (1.99)$$

where in the definition of eikonal $\delta(s, b)$ one has to integrate in direction of the projectile particle.

Now, we are in situation, that the potential $V(r)$ consists of Coulomb part $V_C(r)$ and hadron part $V_H(r)$. And since the relation between potential and eikonal is linear we obtain

$$V(r) = V_C(r) + V_H(r), \quad \delta(s, b) = \delta_C(s, b) + \delta_H(s, b), \quad (1.100)$$

where $\delta_{C,H}(s, t)$ corresponds to $V_{C,H}(r)$ respectively. The last equation can be written in terms of amplitudes and it turns out to have form of (1.98) with

$$R(s, t) = \frac{i}{2\pi pW} \int_\Omega d^2\mathbf{q}' T_C(s, -q'^2) T_H(s, -|\mathbf{q} - \mathbf{q}'|^2). \quad (1.101)$$

We used $\mathbf{q}'$ for a two dimensional vector with magnitude q' and integration region Ω is circle with center at origin and radius q_{max} . Similarly, $\mathbf{q}$ is two dimensional vector with magnitude $q = \sqrt{-t}$ and with arbitrary direction.

The next step is to determine the Coulomb amplitude. There is a complication caused by zero mass of photon. A standard treatment is to work with fictitious mass λ and in the final formula apply limit $\lambda \rightarrow 0$. This way is used in [23] with result

$$T_C(s, -q^2) = \pm \frac{\alpha s}{q^2 + \lambda^2} e^{\mp i\alpha\eta(q)}, \quad \eta(q) = \ln \frac{\lambda^2}{q^2}. \quad (1.102)$$

It differs from the first Born approximation only by presence of λ and phase. Note one cannot apply limit $\lambda \rightarrow 0$ directly to this formula, since the limit of $\exp(i\eta(q))$ is not defined. One can plug form factors for both particles to the Coulomb amplitude. We will assume both form factors are the same and then the Coulomb amplitude is given

¹²⁾ It is a high energy approximation. Hence, it assumed that the wave function $\psi(\mathbf{x})$ does not differ too much from plane wave $e^{i\mathbf{x}\cdot\mathbf{k}}$, where $\mathbf{k}$ is momentum of the particle. In other words, function $\varphi(\mathbf{x})$, $\psi(\mathbf{x}) = \varphi(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}}$, should be slowly varying. Then one can use approximation $\nabla\varphi \ll \mathbf{k}$.

¹³⁾ Actually, formula (1.99) is taken from [22]. The authors did not use the standard eikonal approximation but derived a series of exact equations for function $V(s, r)$. But their only argument for interpretation of $V(s, r)$ as a potential is that Eq. (1.99) is just the same as in eikonal approximation.

by (1.102) multiplied by $F_C^2(t)$. Substituting it to (1.101) and following [23] it yields

$$T(s, t) e^{\pm i\alpha\eta(q)} = \mp \frac{\alpha s}{t} F_C^2(t) + T_H(s, t) \left[\int_{t_{min}}^0 dt' \left(\frac{t}{t'} \right)^{\mp i\alpha} \frac{dF_C^2(t')}{dt} - \right. \\ \left. - \frac{\pm i\alpha}{2\pi} \int_{t_{min}}^0 dt' \frac{F_C^2(t')}{t'} \left(\frac{t}{t'} \right)^{\mp i\alpha} \int_0^{2\pi} d\varphi \left(\frac{T_H(s, t'')}{T_H(s, t)} - 1 \right) \right], \quad (1.103)$$

$$t'' = -|\mathbf{q} - \mathbf{q}'|^2 = t + t' + 2\sqrt{tt'} \cos \varphi. \quad (1.104)$$

It this form, all the pathological dependence on λ is contained in the exponential factor on l.h.s. And this factor vanishes when one computes differential cross section. To derive this form, one has to assume general properties of form factor $F(t = 0 \text{ GeV}^2) = 1$ and $F(t_{min}) \approx 0$. $s/2pW \approx 1$ is to be adopted as well. The next step in [23] is expansion of the power of t/t' to Taylor series

$$\left(\frac{t}{t'} \right)^{\mp i\alpha} = 1 \mp i\alpha \ln \frac{t}{t'} + \dots \quad (1.105)$$

The first two terms are taken into account in the first integral in (1.103), in the second integral only one term of expansion is said to contribute. We are a little suspicious with this simplification because it is valid only for $\alpha \ln t/t' \approx 0$ and if t' tends to 0 (boundary of integration region), the logarithm diverges. But when one presses on, obtained result agrees with result of the Feynman diagrams technique. It seems this discrepancy is almost harmless.

Authors of [23] continue making simplifications and that is why we stop following their paper and make the last modification as in paper [20]. It can be schematically written down

$$\int_{t_{min}}^0 dt' \int_0^{2\pi} d\varphi = 2 \int_{\Omega} d^2 \mathbf{q}' = 2 \int_{\Omega'} d^2 (\mathbf{q} - \mathbf{q}') \approx \int_{t_{min}}^0 dt'' \int_0^{2\pi} d\varphi. \quad (1.106)$$

The region Ω' is circle with radius q_{max} and with center $\mathbf{q}$. Hence, it differs from Ω only by shift by $\mathbf{q}$. Furthermore, at present experiments value of q is such that

$$q \ll q_{max}, \quad (1.107)$$

thus the difference between Ω and Ω' can be neglected. Putting all together (and dropping the irrelevant phase factor on l.h.s.) one obtains

$$T(s, t) = \mp \frac{\alpha s}{t} F_C^2(t) + T_H(s, t) \left[1 \pm i\alpha \int_{t_{min}}^0 dt' \left(\ln \frac{t'}{t} \frac{dF_C^2(t')}{dt} - \right. \right. \\ \left. \left. - \left(\frac{T_H(s, t')}{T_H(s, t)} - 1 \right) I(t, t') \right) \right], \quad (1.108)$$

$$I(t, t') = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{F_C^2(t'')}{t''}. \quad (1.109)$$

For simplicity we will use the dipole form factor and corresponding derivative

$$F_C(t) = \left(\frac{\Lambda}{\Lambda - t} \right)^2, \quad \frac{dF_C^2}{dt} = \left(\frac{\Lambda}{\Lambda - t} \right)^4 \frac{4}{\Lambda - t}, \quad (1.110)$$

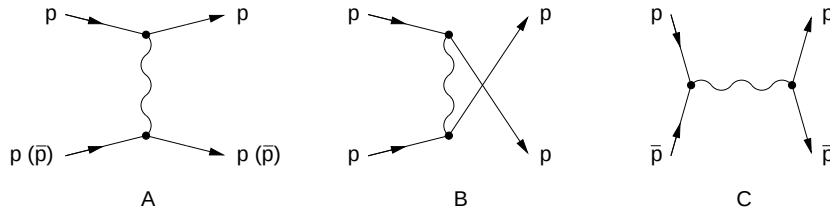
with $\Lambda = 0.71 \text{ GeV}^2$.

Formula (1.108) has two directions of use. First, it can be used to predict differential cross section when we have a model for the hadron amplitude. Or reversely, it can be used for extraction of the hadron amplitude from experimental data.

For completeness, let us give a brief review of the second approach based on Feynman diagrams technique. For any details we refer the reader to Refs. [24], [25] and [26]. Authors of the papers make a strong simplification from the very beginning. They describe proton (and antiproton) by a complex scalar field, not by a Dirac field as it is adequate for spin 1/2 particles. However, it makes all calculations much easier. And in the former approach, spin of particles is not considered at all.

First, let us compute pure electromagnetic amplitude $T_C(s, t)$. The interaction lagrangian can be obtained by the standard way. The derivatives in kinetic term of free field are replaced by the covariant derivatives. This prescription gives two interaction terms. But as we want to retain analogy with fermion QED, we take only the three-leg vertex into account. Then the lowest non-vanishing graphs are shown in Fig. 1.24. Graphs A and B are valid for pp scattering and graphs A and C for $\bar{p}p$ scattering. Indeed, corresponding amplitudes are linear in α .

Fig. 1.24: The lowest order Feynman graphs for pure electromagnetic scattering.



Now we will present a very simple heuristics and show that major contribution in high energy and small angle scattering comes from graph A. This graph contains factor $1/t$ from photon propagator but graph B (resp. C) contains factor $1/u$ (resp. $1/s$). The variable $u = 4M^2 - s - t$ is the third Mandelstam variable. Furthermore, interesting physics (for the elastic scattering) occurs at small t region, i.e., region where

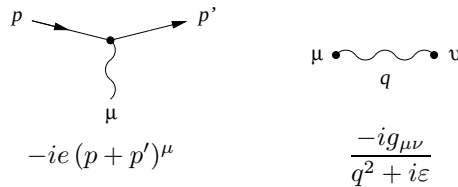
$$s \gg |t|. \quad (1.111)$$

This implies

$$u \approx -s \text{ and } \frac{1}{|t|} \gg \frac{1}{|u|} \quad (1.112)$$

and that is why the graph A dominates the graph B or C.

Fig. 1.25: Feynman rules for scalar QED.



In Fig. 1.25 we showed some of Feynman rules for scalar QED (for details see Ref. [5]). The presented vertex rule holds for particles with four-momenta p and p' . For antiparticles with the same momenta, the corresponding expression has opposite sign, i.e., $ie(p + p')^\mu$. Then, straightforward application of these rules on graph A in Fig. 1.24 leads to invariant amplitude

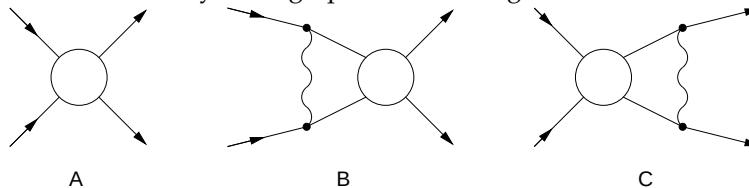
$$|\mathcal{M}_C(s, t)| = \frac{e^2(s - u)}{|t|}, \quad (1.113)$$

where $e = \sqrt{4\pi\alpha}$ is electric charge of proton. With the help of Eqs. (1.112) and (1.28) it can be recast in terms of T amplitude

$$|T_C(s, t)| = \frac{\alpha s}{|t|}, \quad (1.114)$$

which is in accordance with fermion QED, Born amplitude and the eikonal amplitude in the previous approach.

Fig. 1.26: The lowest order Feynman graphs contributing to Coulomb-hadron interference.



If we want to calculate Coulomb-hadron interference with use of Feynman diagrams, we need to represent the hadron amplitude $T_H(s, t)$ by a graph. We will use graph A in Fig. 1.26 for this purpose. But we immediately encounter a big problem. If this graph is a part of a larger graph, the four-momenta attached to lines of graph A may be arbitrary. In other words they do not satisfy the mass shell condition for proton (resp. antiproton) and we cannot use the (Islam's) hadron amplitude since it is defined only on the mass shell. We will return to this problem in a sequel. But now we would like to present next step in [24]. Their calculation is based on graphs B and C from Fig. 1.26. But the zero photon mass gives birth of IR divergences of the mentioned graphs. Authors of [24] gave up evaluating complete graphs and introduced the model independent contribution instead. It does not suffer from the divergences and depend on hadron amplitude on the mass shell only. Their resulting formula reads

$$T(s, t) = T_C(s, t) e^{i\alpha\varphi(s, t)} + T_H(s, t), \quad (1.115)$$

$$\varphi(s, t) = -2 \ln \frac{\sqrt{-t}}{2p} + \int_{t_{min}}^0 \frac{dt'}{|t' - t|} \left(1 - \frac{T_H(s, t')}{T_H(s, t)} \right). \quad (1.116)$$

It is shown in [23] that this formula can be obtained also within the eikonal approach. One has to omit form factors and make several additional approximations.

In [24], one more approximative step was made. The hadron amplitude was parameterized

$$T_H(s, t) \sim \exp(a + bt) , \quad (1.117)$$

which was perfectly acceptable at that times. Performing the integration in Eq. (1.116) one obtains¹⁴⁾

$$\varphi(s, t) \approx \pm \left(\gamma + \ln \frac{-B(s, 0)t}{2} \right) . \quad (1.118)$$

Actually, we mentioned a generalization valid for $\bar{p}p$ too. The function $B(s, t)$ is the diffraction slope defined by Eq. (1.40).

This approximate formula is used for data analysis very often, until nowadays, although the parameterization (1.117) is no more admissible. The comparison of formulae (1.108) and (1.118) is shown at Figs. 1.27, 1.29 and 1.31 and Table 1.3. In this table, one can find differential cross sections for pure hadron scattering, pure Coulomb scattering and full differential cross section obtained by formulae (1.108) and (1.118). In the last column there is relative difference between the mentioned formulae. More precisely it is difference between the 4th and 5th column divided by the 4th column. For each process, there are 3 rows corresponding to different values of t . The first row stands for region where the Coulomb amplitude dominates, the last row for region where the hadron amplitude dominates. The middle row represents region where both amplitudes are comparable.

Table 1.3: Comparison of formulae (1.108) and (1.118).

$-t$ (GeV ²)	$d\sigma/dt$ (mb/GeV ²)				difference (%) of (1.108) and (1.118)
	hadron	coulomb	full (1.108)	full (1.118)	
pp at energy of 53 GeV					
0.001	93.5	258	319.01	317.92	0.34
0.003	91.0	28.0	108.56	108.37	0.17
0.010	82.8	2.33	81.79	81.89	−0.12
$\bar{p}p$ at energy of 541 GeV					
0.001	202	257	501.10	484.62	3.3
0.004	192	15.6	215.79	213.54	1.0
0.015	161	0.980	162.44	162.53	−0.058
pp at energy of 14 TeV					
0.0005	609	1040	1470	1400	4.5
0.0015	590	114	648	630	2.8
0.0060	513	6.77	509	505	0.65

Table 1.3 shows that the difference between formulae (1.108) and (1.118) is much smaller for pp scattering at $W = 53$ GeV than for other processes. This can be explained

¹⁴⁾ Actually, formula (1.118) was first derived in paper [26].

by the fact that the mentioned process is better described by (1.117) than the other processes. The parameterization (1.117) is admissible between $t = 0 \text{ GeV}^2$ and the diffraction dip. In this region the differential cross section falls from 10^2 to almost 10^{-5} mb/GeV^2 . For the remaining processes this fall is lower than 5 orders. Moreover, the diffraction dip is closer to $t = 0 \text{ GeV}^2$ and thus the region where (1.117) holds, is smaller.

At Figs. 1.28, 1.30 and 1.32 we plotted relative importance of the interference term $R(s, t)$ (see formula (1.98)). We represented this importance by fraction

$$f(s, t) = \frac{|T(s, t)|^2 - |T_H(s, t)|^2 - |T_C(s, t)|^2}{|T_H(s, t)|^2}. \quad (1.119)$$

These figures indicate that the Coulomb–hadron interference term cannot be neglected even for higher $|t|$ values as it is commonly done.

Figs. 1.27–1.30 were plotted with the use of parameters obtained from our fits (see Tables 1.1 and 1.2).

Fig. 1.27: Differential cross sections for pp scattering at energy of 53 GeV. The blue curve was obtained by using formula (1.108) while the red by approximate formula (1.118).

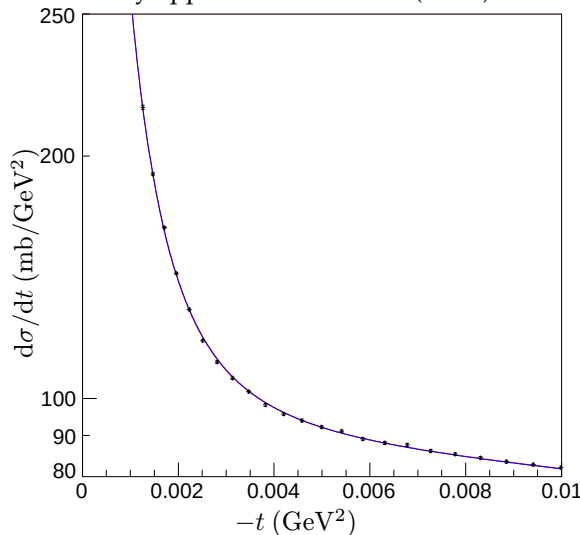


Fig. 1.28: Relative contribution of the interference term for pp scattering at energy of 53 GeV.

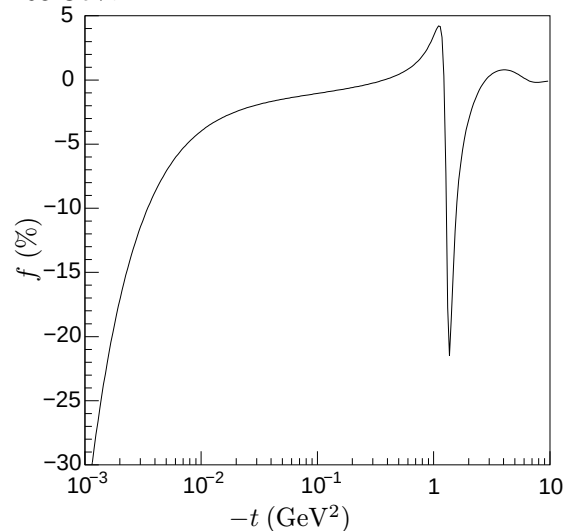


Fig. 1.29: Differential cross sections for $\bar{p}p$ scattering at energy of 541 GeV. The blue curve was obtained by using formula (1.108) while the red by approximate formula (1.118).

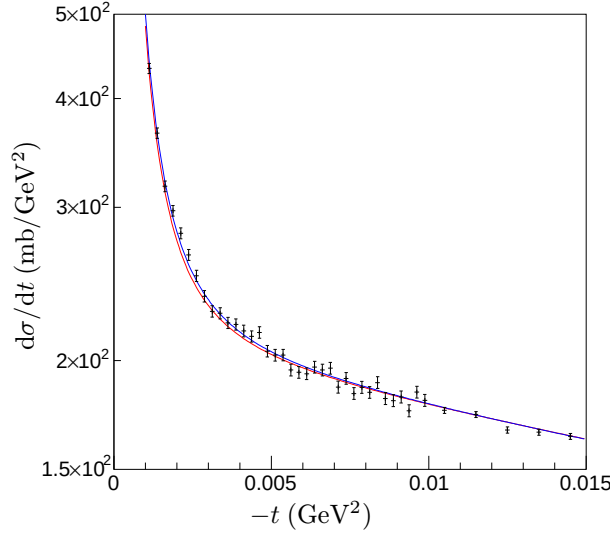


Fig. 1.30: Relative contribution of the interference term for $\bar{p}p$ scattering at energy of 541 GeV.

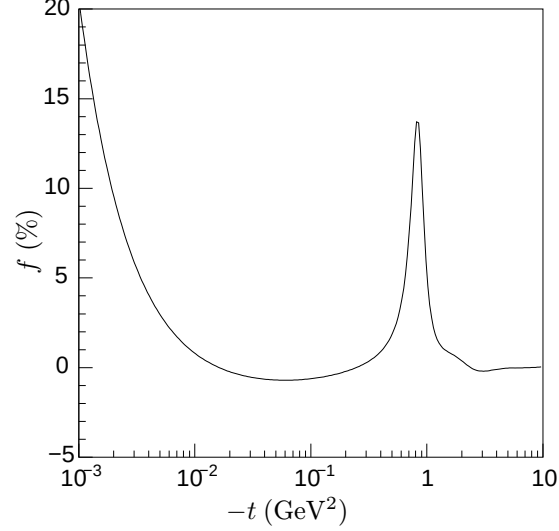


Fig. 1.31: Differential cross sections for pp scattering at energy of 14 TeV. The blue curve was obtained by using formula (1.108) while the red by approximate formula (1.118).

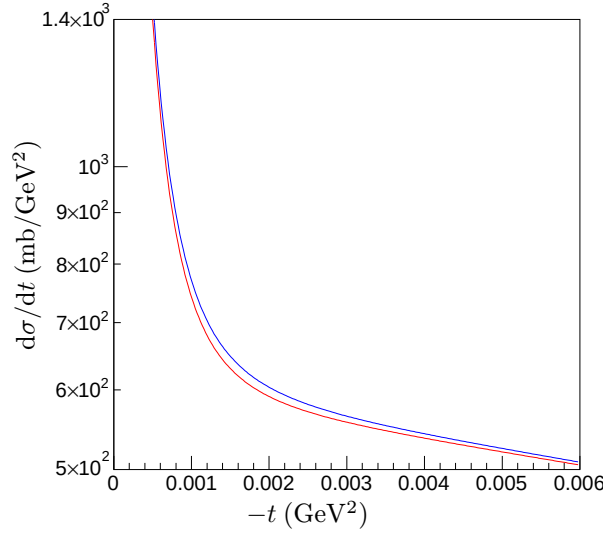
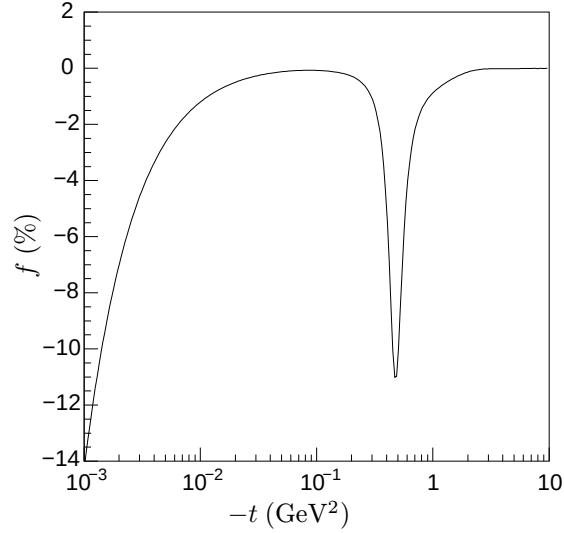
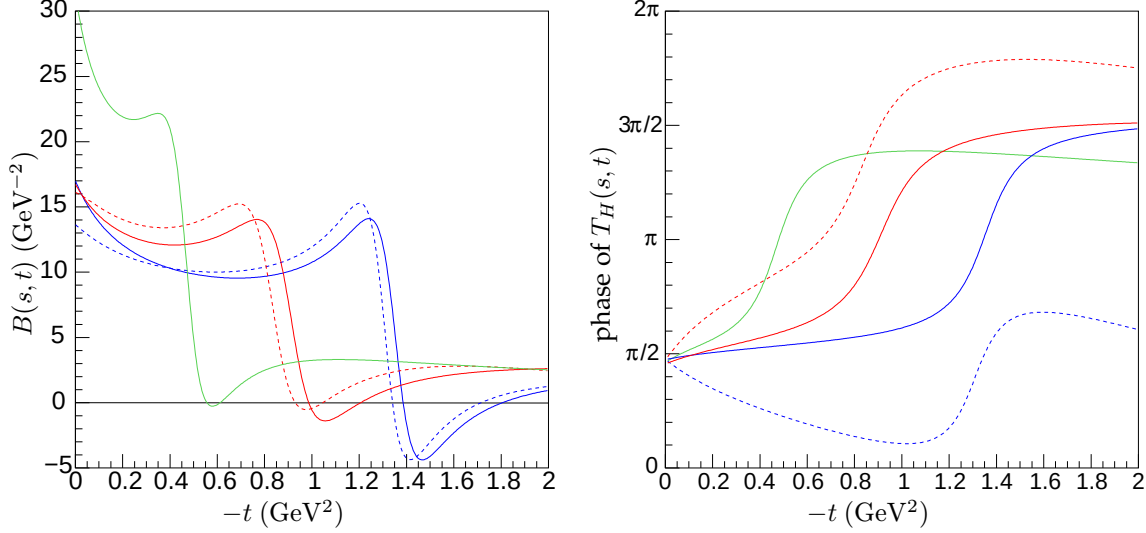


Fig. 1.32: Relative contribution of the interference term for pp scattering at energy of 14 TeV.



The assumption (1.117) requires diffraction slope $B(s, t)$ and phase of the hadron amplitude to be constant for all values of t . Prediction of Islam's model for these functions is shown in Fig. in 1.33. Both functions vary significantly which is in contradiction with Eq. (1.117). While the phase can be different in every model, the diffraction slope is determined by experimental data and thus it must be (almost) same in all models. In other words, the experimental data rule out approximation (1.117).

Fig. 1.33: Diffraction slope $B(s, t)$ and phase of the hadron amplitude $T_H(s, t)$. pp scattering at energy of 53 GeV is drawn blue, $\bar{p}p$ scattering at energy of 541 GeV is drawn red and pp scattering at energy of 14 TeV green. The solid curves were obtained using parameters from papers [6] and [7]. The dashed curves represent our fits, see Tables 1.1 and 1.2.



1.5. Computation

So far we have used natural units, i.e., convention $\hbar = c = 1$. For practical calculations a different choice of units may be more convenient. For instance, we wanted to obtain differential cross section in mb/GeV^2 and impact parameter b in fm. Therefore we had to plug $\hbar$ and c constants appropriately to our formulae. Eqs. (1.28) and (1.29) need to be replaced with

$$\frac{d\sigma}{dt} = \frac{\pi (\hbar c)^2}{s p^2} |T(s, t)|^2, \quad \sigma_{tot}(s) = \frac{4\pi (\hbar c)^2}{pW} \Im T(s, 0). \quad (1.28', 1.29')$$

Similarly, the impact parameter b must be replaced by fraction $b/\hbar c$. For example formulae (1.62) or (1.76) become

$$U(s, q) = \frac{\xi}{(\hbar c)^2} \int_0^\infty b db J_0\left(\frac{bq}{\hbar c}\right) a(s, b) + \frac{\xi}{(\hbar c)^2} \int_0^\infty b db J_0\left(\frac{bq}{\hbar c}\right) \tilde{a}(s, b), \quad (1.62')$$

$$\langle b^2(s) \rangle_{tot} = 4 (\hbar c)^2 \left. \frac{d}{dt} \ln \Im T(s, t) \right|_{0_-}. \quad (1.76')$$

We used numerical values

$$\hbar c \doteq 0.197327 \text{ GeV fm}, \quad (1.120)$$

$$(\hbar c)^2 \doteq 0.389379 \text{ GeV}^2 \text{ mb} = 0.0389379 \text{ GeV}^2 \text{ fm}^2. \quad (1.121)$$

All our numerical calculations were done with the help of the ROOT system. It is a data analysis system developed in CERN. It is object-oriented and it is based on

C++ and partly on FORTRAN CERN libraries. We always used double precision type (Double_t). For numerical integration we used method TF1::Integral, which exploits 8 and 16-point Gaussian quadrature approximations. During tests it turned out to be very reliable. However, in some cases manual intervention was necessary. For instance in the case of Eq. (1.108). The problematic part is the last term in square brackets. If $t' \rightarrow t$ integral $I(t, t')$ diverges while the subtraction in parentheses tends to 0. Both factors together are finite. To avoid troubles we used following trick. We split the integral

$$\int_{t_{min}}^0 \left(\frac{T_H(t')}{T_H(t)} - 1 \right) I(t, t') = \int_{t+\tau}^0 \left(\frac{T_H(t')}{T_H(t)} - 1 \right) I(t, t') + \int_{t-T}^{t-\tau} \left(\frac{T_H(t')}{T_H(t)} - 1 \right) I(t, t') , \quad (1.122)$$

where the equality holds if $\tau \rightarrow 0$ and $t - T \rightarrow t_{min}$. If one takes τ finite but small, deviation from (1.122) will be small as well. Simultaneously, one avoids problems with $I(t, t')$ divergence. We made several tests for values of τ round 10^{-4} GeV^2 . And differences of values of the discussed integral were negligible. Therefore we fixed τ at the mentioned value. Similarly, we can raise lower bound for the second integral in (1.122). Since $I(t, t')$ falls quickly when t' draws apart t , the contribution to the integral from region of t' far from t is negligible. That is why we consider the lower bound in form $t - T$. T denotes size of t' region which is taken into account. Again, we made several tests and for values around $T = 10 \text{ GeV}^2$ and higher the variation of the integral value was negligible.

References

- [1] V. Barone, E. Predazzi, High-Energy Particle Diffraction, *Springer* (2002)
- [2] J. Formánek, Úvod do relativistické kvantové mechaniky a kvantové teorie pole, *Karolinum* (2000)
- [3] J. Hořejší, Fundamentals of electroweak theory, *Karolinum* (2002)
- [4] A. Martin, *Phys. Lett.* **B404** (1997) 137–140
- [5] M. E. Peskin and D. V. Schroeder, An Introduction to quantum field theory, *Addison–Wesley* (1995)
- [6] M. M. Islam, V. Innocente, T. Fearnley and G. Sanguietti, *Europhys. Lett.* **4** (1987) 189–196
- [7] M. M. Islam, R. J. Luddy and A. V. Prokudin, *Mod. Phys. Lett. A* **18** (2003) 743–752
- [8] M. M. Islam, R. J. Luddy and A. V. Prokudin, *Phys. Lett.* **B605** (2005) 115–122
- [9] M. M. Islam, *Z. Phys.* **C53** (1992) 253–262
- [10] M. M. Islam, *Found. Phys.* **24** (1994) 419–425
- [11] Landolt–Börnstein Series, Vol. 1, *Springer* (1980)
- [12] M. Bozzo et al., *Phys. Lett.* **B147** (1984) 385
- [13] C. Augier et al., *Phys. Lett.* **B316** (1993) 498
- [14] D. Bernard et al., *Phys. Lett.* **B171** (1986) 142

-
- [15] T. Adachi, T. Kotani, *Progr. Theor. Phys. Suppl.* **Commemoration issue** (1965) 316–331
 - [16] T. Adachi, T. Kotani, *Progr. Theor. Phys. Suppl.* **37, 38** (1966) 297–305
 - [17] V. Kunderát, M. Lokajíček and D. Krupa, *Phys. Lett.* **B544** (2002) 132–138
 - [18] Eric W. Weisstein, Bessel Function of the First Kind, *From MathWorld—A Wolfram Web Resource* <http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html>
 - [19] V. Kunderát and M. Lokajíček Jr. and M. Lokajíček, *Czech. J. Phys.* **B31** (1981) 1334
 - [20] V. Kunderát and M. Lokajíček, *Z. Phys.* **C63** (1994) 619–630
 - [21] M. M. Islam, *Nucl. Phys.* **B104** (1976) 511
 - [22] M. M. Islam, High energy physics and fundamental particles, *Gordon and Breach, science publishers*, volume XB (1967)
 - [23] R. Cahn, *Z. Phys.* **C15** (1982) 253
 - [24] G. B. West and D.R. Yennie, *Phys. Rev.* **172** (1968) 1413–1422
 - [25] J. Rix, R. M. Thaler, *Phys. Rev.* **152** (1966) 1357–1363
 - [26] M. P. Locher, *Nucl. Phys.* **B2** (1967) 525

Chapter 2

Detector alignment

As indicated in the introduction the TOTEM experiment will exploit a special detection technique that is able to detect particles scattered to very small angles. The basic elements of this technique are the Roman pots and edgeless detectors. The Roman pots are devices that can move the detectors near and far from the beam. The detectors stay far from the beam until it gets stable and of well defined shape and thickness. Then the detectors will be moved to the beam so as their edge will be about 1 mm apart from the beam axis.

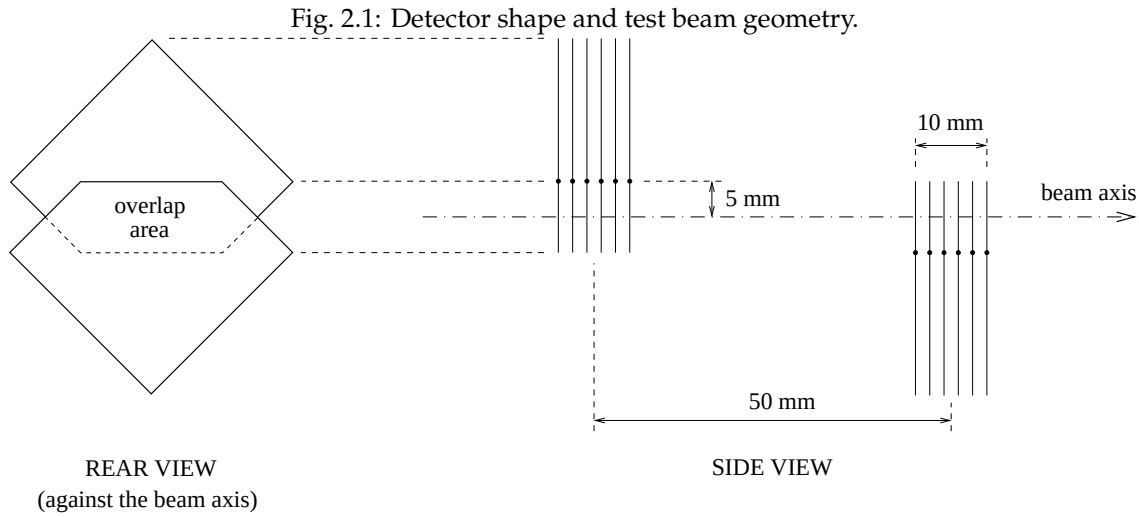
The detectors inside the Roman pots will be planar silicon strip detectors with strip pitch $66\text{ }\mu\text{m}$. In each Roman pot, the detectors will be divided to two groups such that strips in one group will be perpendicular to strips of the other group. Common strip detectors have at their edges special voltage terminating structures. These structures improve stability, but are insensible. This is in contradiction with our physical aim — to detect particles as close to the beam as possible. That is why special edgeless strip detectors are developed for TOTEM experiment (see [27]).

These detectors have to be tested. For this purpose we used test beam facility at X5 at CERN. It is based on SPS accelerator that accelerates protons. The protons are extracted and hitting a target they produce pions at most. Pions decay into muons. Thanks to different muon and pion interaction with matter, after the beam passes obstacle wide enough (pion dump), there are only muons in the beam. And this muon beam was used to test the detectors.

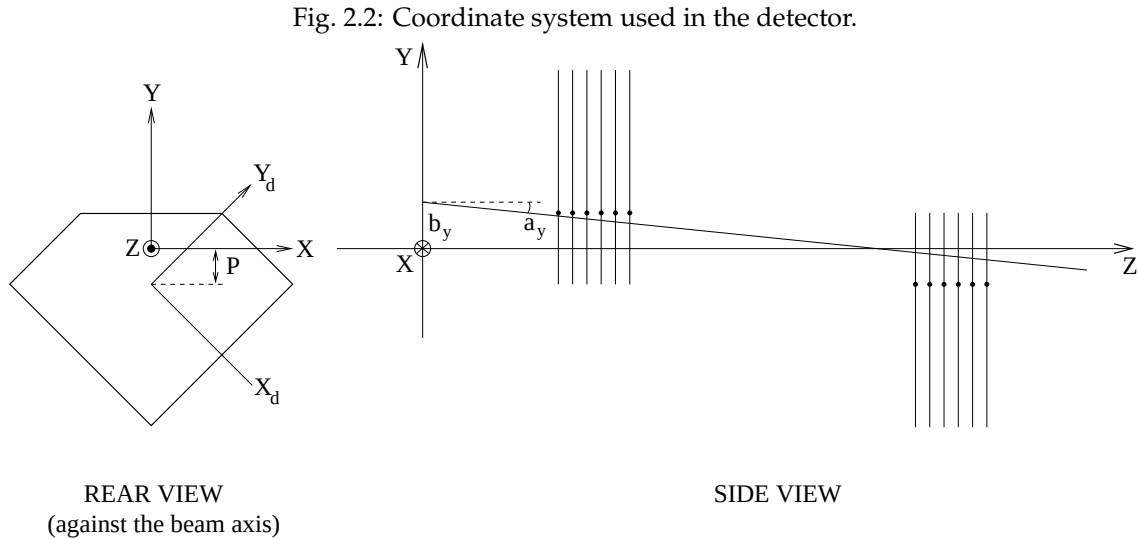
We are interested to probe especially the edge area of the detectors, which is very small. Thus we will have to fix detector positions as precisely as possible. However, there will always be some mechanical displacement. Our aim in this chapter is to develop a method how to acquire maximum information about the detector displacements from test beam data. Our approach is based the fact we know some statistical properties of test beam.

2.1. Testing edgeless strip detectors

Proportional sketch of the test beam detector configuration is shown at Fig. 2.1. There are 2 packages per 6 detectors. All the detectors are used for track reconstruction and testing simultaneously.



As the magnetic field is negligible in the detector area the particle tracks inside the detector will be the straight lines. To describe the tracks we will use reference frame shown at the right side of Fig. 2.2, where we chose the axis z as the beam axis. A track within this frame can be determined by slopes and offsets in $Z - X$ and $Z - Y$ plane. At the Fig. 2.2 there is a sample track with shown offset b_y and slope a_y (by definition tangent of corresponding angle) in $Z - Y$ plane¹⁵⁾.



For description of hits in individual planar detectors it is natural to use coordinate system with subscript d as it is shown on the left part of Fig. 2.2. The axes are parallel to strips and system origin is placed in the geometrical center of the detector (if we omit the cut corner).

¹⁵⁾ Similarly slope and offset in $Z - X$ plane will be denoted a_x and b_x

For further discussion it is essential to know statistical behavior of tracks in the test beam. Due to technical construction of the beam source we can expect slope a (in both planes) to have approximately Gaussian distribution with mean value $\langle a \rangle$ and standard deviation σ_a

$$\langle a \rangle = 0, \quad \sigma_a = 3 \cdot 10^{-3}. \quad (2.1)$$

Both offsets b_x and b_y have approximately uniform distribution. The size of the beam source is of order 10 cm. But during experiment we will detect only particles that pass through trigger. It effectively reduces width of the beam to about 4 cm. The latter size induces standard deviation

$$\sigma_b = \frac{4 \text{ cm}}{\sqrt{12}} \approx 12 \text{ mm}. \quad (2.2)$$

So far we have described ideal case. But we are able to assemble the detector with finite precision only. Error of mechanical placement of detectors is estimated to be less than

$$\Delta = 20 \text{ } \mu\text{m}. \quad (2.3)$$

These error shifts can cause the detectors to slant. Maximum slant is produced when one end of detector will be shifted in one direction and opposite end to opposite direction. One can expect maximum slant

$$s = \frac{2\Delta}{L} = 1 \cdot 10^{-3} \quad (2.4)$$

where $L = 4 \text{ cm}$ as a detector size was used.

2.2. Track reconstruction

Due to the displacements of detectors it is necessary to distinguish for all quantities *measured* values (values acquired by our apparatus) and *actual* values (values that would be measured if there were no shifts and slats). That is why we will use the following convention. Actual values will be denoted without primes (e.g., for actual y hit position in the i -th detector we will use y_i), measured values will be denoted with primes (y'_i).

As the particle tracks are linear we can employ linear regression for reconstruction. Let us use x'_i or y'_i for (X or Y) coordinate in the i -th detector (in accordance with which coordinate the i -th detector measures). Then the standard linear regression formulas for slope a'_x and offset b'_x plane read

$$a'_x = \frac{1}{\mathcal{D}}(S_{zx}S_1 - S_xS_z), \quad (2.5)$$

$$b'_x = \frac{1}{\mathcal{D}}(S_{zz}S_x - S_{zx}S_z), \quad (2.6)$$

where we used useful abbreviations

$$S_1 = \sum_i \frac{1}{\sigma^2}, \quad S_z = \sum_i \frac{z_i}{\sigma^2}, \quad S_{zz} = \sum_i \frac{z_i^2}{\sigma^2}, \quad (2.7)$$

$$S_x = \sum_i \frac{x'_i}{\sigma_i^2}, \quad S_{zx} = \sum_i \frac{z_i x'_i}{\sigma_i^2} \quad \text{and} \quad \mathcal{D} = S_{zz}S_1 - S_zS_z. \quad (2.8)$$

The sums go over all detectors measuring X coordinate that are active in the event. z_i stands for Z position of the i -th detector and σ_i stands for measurement error in this detector.

Besides a'_x and b'_x the linear regression can give us the covariance matrix corresponding to these quantities. Using the abbreviations (2.7) and (2.8) the covariance matrix can be written as

$$-\frac{1}{\mathcal{D}} \begin{pmatrix} S_1 & S_z \\ S_z & S_{zz} \end{pmatrix}. \quad (2.9)$$

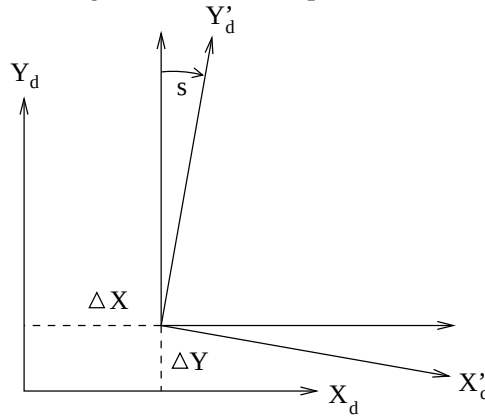
The similar equations can be used for track reconstruction in $Z - Y$ plane, only replacing x by y .

Note that a'_x and a'_y need not be the same as a_x and a_y (due to the detector shifts and slants).

2.3. Calibration

As it was mentioned above we can expect two types of displacement for every detector – shift perpendicular to the beam axis and slant (small rotation around the beam axis). The situation is depicted at Fig. 2.3.

Fig. 2.3: Detector displacement.



Non-displaced detector has coordinate system $X_d - Y_d$. After shifting the detector by Δx resp. Δy in X resp. Y direction and rotating by angle s we receive primed system $X'_d - Y'_d$ of displaced detector.

Slant estimation for s is of order 10^{-3} . In this region of values we can put

$$\cos s \approx 1, \quad \sin s \approx s. \quad (2.10)$$

Then, the relation between actual hit position and position measured by i -th displaced detector is given by

$$\begin{pmatrix} x'_i \\ y'_i \end{pmatrix} = \begin{pmatrix} 1 & -s_i \\ s_i & 1 \end{pmatrix} \begin{pmatrix} x_i - \Delta x_i \\ y_i - \Delta y_i \end{pmatrix} = \begin{pmatrix} x_i - \Delta x_i - (y_i - \Delta y_i)s_i \\ y_i - \Delta y_i + (x_i - \Delta x_i)s_i \end{pmatrix}. \quad (2.11)$$

Having obtained these relations one can also derive relations between actual and measured track parameters. Let us note that Eqs. (2.5) and (2.6) describing the track reconstruction method are linear in x' . Therefore the relations for measured slopes and offsets must have form

$$\begin{pmatrix} a'_x \\ a'_y \end{pmatrix} = \begin{pmatrix} a_x - \Delta a_x - \delta a_x \\ a_y - \Delta a_y + \delta a_y \end{pmatrix}, \quad \begin{pmatrix} b'_x \\ b'_y \end{pmatrix} = \begin{pmatrix} b_x - \Delta b_x - \delta b_x \\ b_y - \Delta b_y + \delta b_y \end{pmatrix}, \quad (2.12)$$

where the terms with Δ denote perturbations due to the shifts and the terms with δ stand for the slant perturbations. We will keep this notation in what follows. Performing the calculation one can identify (for example for a'_x part)

$$\Delta a_x = \frac{1}{\mathcal{D}}(S_z \Delta x S_1 - S_z S_{\Delta x}), \quad \delta a_x = \frac{1}{\mathcal{D}}(S_z y_s S_1 - S_z S_{y_s}), \quad (2.13)$$

where

$$\begin{aligned} S_{\Delta x} &= \sum_i \frac{\Delta x_i}{\sigma^2}, \quad S_z \Delta x = \sum_i \frac{z_i \Delta x_i}{\sigma^2}, \\ S_{y_s} &= \sum_i \frac{(y_i - \Delta y_i) s_i}{\sigma^2}, \quad S_z y_s = \sum_i \frac{z_i (y_i - \Delta y_i) s_i}{\sigma^2} \end{aligned} \quad (2.14)$$

and other abbreviations are given by Eqs. (2.7) and (2.8). The other symbols in (2.12) have very analogical meanings as $\Delta a_x, \delta a_x$ in (2.13).

Now we are going to employ knowledge of distribution of track in the beam. Crucial point is having beam with zero mean slope (see Eq. (2.1)). For convergence of this method we also need zero mean offset of the tracks. Here we will assume these conditions are met. Later we will discuss their fulfilling including influence of detector shape.

We will make one more simplification here. We will pretend the laboratory system $X - Y$ and detector system $X_d - Y_d$ are equivalent. In the next sections we will generalize our ideas to the situation shown at Fig. 2.2.

Mathematical description of our calibration method comes out from Eq. (2.11). For simplicity we will go on discussing X detectors (i.e. detectors measuring X coordinate) only. Let us denote difference between measured positions in i -th and j -th detector D_{ij}

$$D_{ij} = x'_i - x'_j = (x_i - x_j) - (\Delta x_i - \Delta x_j) - (y_i s_i - y_j s_j) + (\Delta y_i s_i - \Delta y_j s_j). \quad (2.15)$$

If we use standard notation for actual track parameters

$$x_i = a_x z_i + b_x, \quad y_i = a_y z_i + b_y, \quad (2.16)$$

we can rewrite Eq. (2.15) as

$$D_{ij} = a_x(z_i - z_j) - (\Delta x_i - \Delta x_j) - a_y(z_i s_i - z_j s_j) - b_y(s_i - s_j) + (\Delta y_i s_i - \Delta y_j s_j). \quad (2.17)$$

With the help of estimations (2.1) and (2.2) we can make order estimation of terms on r.h.s. of Eq. (2.17). For this purpose let us denote Δz a typical z distance between detector planes. All estimations are shown in Table 2.1.

Table 2.1: Order estimation for terms on r.h.s. of Eq. (2.17).

term	estimation	estimation for $\Delta z = 10^{-1}$ m
$a_x(z_i - z_j)$	$\Delta z 10^{-3}$	10^{-4} m
$\Delta x_i - \Delta x_j$	10^{-5} m	10^{-5} m
$a_y(z_i s_i - z_j s_j)$	$\Delta z 10^{-6}$	10^{-7} m
$b_y(s_i - s_j)$	10^{-5} m	10^{-5} m
$\Delta y_i s_i - \Delta y_j s_j$	10^{-8} m	10^{-8} m

This analysis reads that the last term $\Delta y_i s_i - \Delta y_j s_j$ is negligible compared to the others. And that is why it will be omitted in the following.

So far we have discussed analysis of one event (one track) only. But we can count with ensemble of $N = 10^4 \div 10^5$ events. To make use of this ensemble we introduce event-averaged value of D_{ij} (it will be denoted $\bar{D}_{ij}$)

$$\bar{D}_{ij} = \frac{1}{N} \sum_{n=0}^N D_{ij}^n, \quad (2.18)$$

where N is number of events and upper index n expresses the value of D_{ij} is taken from n -th event. One can substitute (2.17) for D_{ij}^n and obtains

$$\bar{D}_{ij} = -(\Delta x_i - \Delta x_j) + \bar{a}_x(z_i - z_j) - \bar{a}_y(z_i s_i - z_j s_j) - \bar{b}_y(s_i - s_j), \quad (2.19)$$

where

$$\bar{a}_x = \frac{1}{N} \sum_{n=0}^N a_x^n, \quad \text{etc.} \quad (2.20)$$

Factor $\bar{a}_x$ has the form of arithmetic mean and thus its standard deviation is given by standard formula

$$\sigma_{\bar{a}_x} = \frac{\sigma_{a_x}}{\sqrt{N}} \quad (2.21)$$

and similarly for a_y and b_y . In other words last three terms on r.h.s. of Eq. (2.19) will be suppressed by factor $\sqrt{N}$ after averaging. For N high enough term $\Delta x_i - \Delta x_j$ will be the most important one and we can present approximate relation

$$\bar{D}_{ij} \approx -(\Delta x_i - \Delta x_j). \quad (2.22)$$

Error of this relation is given dominantly by the second highest term on r.h.s. of Eq. (2.19). It is the term $\bar{a}_x(z_i - z_j)$ and its value can be estimated

$$\frac{\sigma_{a_x}}{\sqrt{N}} (z_i - z_j). \quad (2.23)$$

In the Table 2.2 there are several error estimates for different number of events N and $(z_i - z_j) = 5 \cdot 10^{-2}$ m (which is the distance between detector packages in the test beam configuration). Value of $\sigma_{a_x} = 3 \cdot 10^{-3}$ is given by Eq. (2.1).

Table 2.2: Estimates of error of relation (2.22).

N	error
$1 \cdot 10^4$	$1.5 \mu\text{m}$
$5 \cdot 10^4$	$0.7 \mu\text{m}$

Let us turn to the slant calibration now. Eq. (2.17) truncated of the last (negligible) term and with added upper index n (that stands for event number) reads

$$D_{ij}^n = -b_y^n(s_i - s_j) - (\Delta x_i - \Delta x_j) + a_x^n(z_i - z_j) - a_y^n(z_i s_i - z_j s_j). \quad (2.24)$$

If we omitted last two terms on r.h.s. of previous formula, we could retrieve coefficient $s_i - s_j$ using linear regression applied on data D_{ij}^n versus b_y^n . The last two terms are not negligible and we cannot omit them easily, however we can treat them as a perturbation. Let us denote T_{ij} the result of the regression suggested above. The perturbation causes

$$T_{ij} = -(s_i - s_j) - T_{a_x} - T_{a_y}, \quad (2.25)$$

where

$$\begin{aligned} T_{a_x} &= (z_i - z_j) \frac{\sum b_y a_x \sum 1 - \sum a_x \sum b_y}{\sum b_y^2 \sum 1 - (\sum b_y)^2}, \\ T_{a_y} &= (z_i s_i - z_j s_j) \frac{\sum b_y a_y \sum 1 - \sum a_y \sum b_y}{\sum b_y^2 \sum 1 - (\sum b_y)^2} \end{aligned} \quad (2.26)$$

and all the sums go through all events and to simplify reading upper indices n were dropped at a_x , a_y and b_y . Let us investigate properties of the fraction in definition of T_{a_x} . We will use symbol F for this fraction and recast it to

$$F = \frac{\frac{1}{N} \sum b_y a_x - \frac{1}{N} \sum a_x \frac{1}{N} \sum b_y}{\frac{1}{N} \sum b_y^2 - (\frac{1}{N} \sum b_y)^2}. \quad (2.27)$$

As the random variables a_x and b_y are independent and the mean value $\langle a \rangle = 0$ one can find out

$$\langle F \rangle = 0 \text{ m}^{-1}. \quad (2.28)$$

Computing standard deviation of term F is more difficult. To simplify this task we fixed the denominator to its mean value $\sigma_{b_y}^2$ and computed the standard deviation only for the numerator. Performing the computation one can obtain

$$\sigma_F \approx \frac{\sigma_{a_x}}{\sigma_{b_y}} \sqrt{\frac{1}{N} - \frac{1}{N^2}} \approx \frac{\sigma_{a_x}}{\sigma_{b_y}} \frac{1}{\sqrt{N}}. \quad (2.29)$$

Note that we will have at least $N = 10^3$ events and that is why we can put the last approximate equality in previous relation.

We made several simulations with different event numbers N to test Eq. (2.29). The input values (beam parameters) were taken from Eqs. (2.1) and (2.2). For every event number we generated 10^4 values of F and made histograms (two of them are shown in Figs. 2.4 and 2.5, values of F are in m^{-1}). Gradually, the histograms were

fitted by Gaussian. The standard deviations acquired from fitting are shown in Table 2.3.

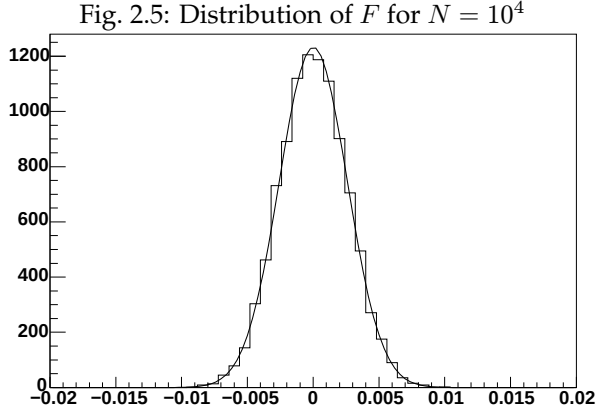
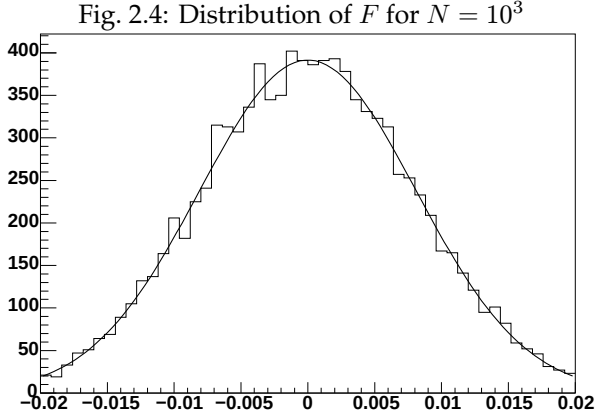


Table 2.3: Test of Eq. (2.29)

N	standard deviation in m^{-1}	
	estimated by Eq. (2.29)	acquired from fitting
$1 \cdot 10^3$	$7.9 \cdot 10^{-3}$	$8.1 \cdot 10^{-3}$
$5 \cdot 10^3$	$3.5 \cdot 10^{-3}$	$3.6 \cdot 10^{-3}$
$1 \cdot 10^4$	$2.5 \cdot 10^{-3}$	$2.5 \cdot 10^{-3}$
$5 \cdot 10^4$	$1.1 \cdot 10^{-3}$	$1.2 \cdot 10^{-3}$

Merging Eqs. (2.26) and (2.29) one can receive estimates for standard deviations

$$\sigma_{T_{ax}} \approx \frac{z_i - z_j}{\sqrt{N}} \frac{\sigma_{ax}}{\sigma_{by}}, \quad \sigma_{T_{ay}} \approx \frac{z_i s_i - z_j s_j}{\sqrt{N}} \frac{\sigma_{ay}}{\sigma_{by}}. \quad (2.30)$$

Therefore for high N terms T_{ax} and T_{ay} become negligible and we can introduce approximate relation

$$T_{ij} \approx -(s_i - s_j). \quad (2.31)$$

Error of the previous relation is given mostly by T_{ax} (see Table 2.1). Several error estimations for different event numbers are evaluated in Table 2.4. For evaluating we used beam parameters summarized in (2.1) and (2.2) and $(z_i - z_j) = 5 \cdot 10^{-2} \text{ m}$ (package distance in test beam configuration).

Table 2.4: Error estimations for Eq. (2.31)

N	error
$1 \cdot 10^4$	$5 \cdot 10^{-4}$
$5 \cdot 10^4$	$2.2 \cdot 10^{-4}$

Let us make several final notes. The first concerns regression process suggested below Eq. (2.24). In practice we do not know actual value of offset b_y . We can only obtain value b'_y by tracking. But the difference between b_y and b'_y should be of order 10^{-5} m (like shifts). Therefore it is negligible in comparison with b_y that is of order 10^{-2} m .

To obtain value of $\Delta x_i - \Delta x_j$ one does not need to use formula (2.22). It can be obtained by regression used to determine $s_i - s_j$. Namely one can use offset obtained

by the regression. Computer simulations show that the difference between these two approaches is much smaller than error estimated by Eq. (2.23).

In the beginning of this part we introduced Eq. (2.15) describing difference between measured positions in two detectors. Consequence of this fact is that we can obtain only shift and slant differences between these detectors, not absolute values. One could try to use relation (2.11) as starting point for analysis and thus use directly x'_i instead of D_{ij} . But then a term containing b_x appears in analog of Eq. (2.17). Even if we knew the mean value of b_x is zero, b_x is of order 10^{-2} m. One can compare this value with values in Table 2.1 and find out it is by factor 100 greater than greatest value in the mentioned table. If we consider that standard deviation of arithmetic mean is proportional to $1/\sqrt{N}$ one can do following guess. Statistics necessary to reduce b_x term to the level of Δx (by averaging analogical to Eq. (2.18)) is approximately $N = 10^6$ events. During experiment we will not be able to collect so many (meaningful) events and that is why we use method suggested in this section.

2.3.1. Calibration in actual configuration

In the previous section we made several simplifications. One of them was using the laboratory coordinate system instead of detector system. From now on we will consider the detector coordinate system as shown on the left side of Fig. 2.2. One can derive relation between coordinates x, y in the laboratory system $X - Y$ and coordinates x', y' in the displaced detector system $X'_d - Y'_d$

$$\begin{pmatrix} x'_i \\ y'_i \end{pmatrix} = \mathcal{R}_{s_i} \left[\mathcal{R}_{\pi/4} \begin{pmatrix} x_i \\ y_i - P_i \end{pmatrix} - \begin{pmatrix} \Delta x_i \\ \Delta y_i \end{pmatrix} \right], \quad (2.32)$$

where $\mathcal{R}_\alpha$ stands for matrix rotating coordinates counter-clockwise by angle α . It reads

$$\mathcal{R}_\alpha = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}.$$

Before tracking and calibration one should unify all the reference frames as much as possible. We tried to eliminate P_i shift by summing

$$\begin{aligned} \begin{pmatrix} x'_i \\ y'_i \end{pmatrix} + \mathcal{R}_{\pi/4} \begin{pmatrix} 0 \\ P_i \end{pmatrix} &= \mathcal{R}_{s_i} \begin{pmatrix} \xi_i - \Delta x_i \\ \eta_i - \Delta y_i \end{pmatrix} + \mathcal{R}_{\pi/4} (1 - \mathcal{R}_{s_i}) \begin{pmatrix} 0 \\ P_i \end{pmatrix} = \\ &= \begin{pmatrix} \xi_i - (\Delta x_i - P_i s_i / \sqrt{2}) - (\eta_i - \Delta y_i) s_i \\ \eta_i - (\Delta y_i - P_i s_i / \sqrt{2}) + (\xi_i - \Delta x_i) s_i \end{pmatrix}, \end{aligned} \quad (2.33)$$

where we used symbols ξ and η for coordinates in reference frame rotated by 45° clockwise from the laboratory one, i.e.

$$\begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} = \mathcal{R}_{\pi/4} \begin{pmatrix} x_i \\ y_i \end{pmatrix}. \quad (2.34)$$

The last equality in (2.33) follows from estimation (2.4) and Eq. (2.10).

We can compare Eqs. (2.33) and (2.11). The first difference is that there are ξ and η in Eq. (2.33) instead of x and y . This interchange only reflects we work in reference frame that is 45° rotated. The second change is presence of term involving P . As the P should be of order of 1 mm, order of the term should be

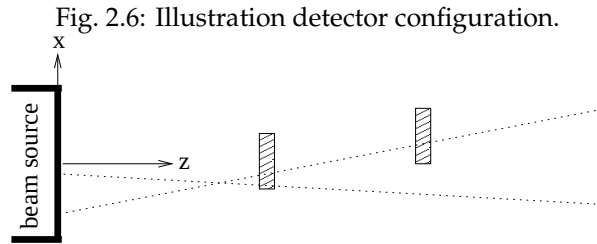
$$P_i s_i \approx 10^{-6}. \quad (2.35)$$

Therefore it cannot be neglected in comparison with the shifts and we have to include this term as part of the shifts. Then the full shift vector reads

$$\begin{pmatrix} \Delta x_i - P_i s_i / \sqrt{2} \\ \Delta y_i - P_i s_i / \sqrt{2} \end{pmatrix}. \quad (2.36)$$

2.3.2. Calibration and detector geometry

We have assumed that our detectors are infinitely large so far. It means that all tracks are detected. In this section we are going to discuss finite size and shape of the detectors.



Let us consider situation drawn in Fig. 2.6 (for simplicity, we restrict the discussion to one plane only, let us say $Z - X$ plane). On the left side you can see the beam source with parameters given by Eqs. (2.1) and (2.2). The two hatched rectangles represent two detectors. Our aim is to describe distribution of slopes and offsets of the tracks detected by these two detectors. It is clear that the slope a and the offset b ¹⁶⁾ cannot have any arbitrary value (see two dotted tracks in Fig. 2.6). The slopes resp. the offsets are confined to intervals (a_{min}, a_{max}) resp. (b_{min}, b_{max}) . In addition the interval of plausible values for the slope is different for different values of the offset. Thus we should properly denote allowed interval for slopes $(a_{min}(b), a_{max}(b))$. The distribution of slopes inside these allowed intervals remains unchanged, but must be renormalized. This is expressed by following relation

$$h(a, b) = \frac{g(b)}{\int_{b_{min}}^{b_{max}} g(b) db} \frac{f(a)}{\int_{a_{min}(b)}^{a_{max}(b)} f(a) da}, \quad (2.37)$$

where $f(a)$ and $g(b)$ are probability distribution functions (p.d.f.) of slopes and offsets for the beam source and $h(a, b)$ is joint p.d.f. of slopes a and offsets b detected by the two detectors.

¹⁶⁾ In this section we will work in $Z - X$ plane only. Therefore we will drop indices x at a_x and b_x .

Let us make two notes here. Formula (2.37) introduces correlation between slopes and slants. Actually even slants and offsets in $Z - X$ and $Z - Y$ planes are correlated for general detector shapes. Thus the fact $\langle a \rangle = 0$ does not imply $\langle F \rangle = 0 \text{ m}^{-1}$, which is a necessary condition for our calibration method explained above.

Geometry of detectors is effectively affected by cuts¹⁷⁾ as well. If one accepts tracks passing through a small area only, it is (from point of view of Eq. (2.37)) the same as the detectors occupied only this small area.

Hence, introducing a desirable cut offers a way to meet condition $\langle F \rangle = 0 \text{ m}^{-1}$. However, evaluating $h(a, b)$ by formula (2.37) may be difficult (even for very symmetric areas) and for instance for Gaussian distributions analytically impossible at all. That is why we describe only qualitatively the geometry influence for our test beam detector configuration.

Let us accept only tracks going through the overlap area (see Fig. 2.1) of the first and the last detector (it is our cut). And assume the detectors are placed in distance D of order 10^{-1} m from the beam source. Let us recall that slope of the tracks should be of order $\sigma_a \approx 10^{-3}$. Therefore, with error $D\sigma_a \approx 10^{-4} \text{ m}$, we can state that interval of allowed offset is given by size of the overlap area. This overlap has typical dimension 1 cm and hence we can make order estimation for $a_{max}(b)$ and $a_{min}(b)$ (actually it is evaluated for $b = 0 \text{ m}$)

$$a_{max}(b) \approx |a_{min}(b)| \approx \frac{1 \text{ cm}}{D} = 10^{-1} . \quad (2.38)$$

It is much more than beam angular spread σ_a and thus we can put

$$\int_{a_{min}(b)}^{a_{max}(b)} f(a) \approx 1 . \quad (2.39)$$

Then we can factorize joint p.d.f. $h(a, b)$ to marginal p.d.fs. for a and b

$$h(a, b) \approx \frac{g(b)}{\int_{b_{min}}^{b_{max}} g(b)} f(a) , \quad (2.40)$$

which means a and b are nearly independent. In addition due to the symmetric shape of overlap area, variables a and b keep their zero mean values

$$\langle a \rangle = 0, \quad \langle b \rangle = 0 \text{ m} . \quad (2.41)$$

It implies

$$\langle F \rangle = 0 \text{ m}^{-1} . \quad (2.42)$$

The conclusion of this section is that calibration method suggested in Section 2.3 can be used also for detectors with finite size. Necessary condition is to use desirable cut. Computer simulations confirm this conclusion.

¹⁷⁾ Here, the word cut means a choice of events (i.e., tracks) which are taken into account.

2.4. Programming

Two ROOT programs were produced. The first simulates detector–beam interaction and the second analyzes acquired data.

The simulation program generates random (beam particle) tracks on basis of formulae (2.1) and (2.2). With the help of Eq. (2.32) positions of hits are calculated. These positions are converted into strip numbers and stored in a ROOT tree. This ROOT tree has the same structure as was used during the test beam. Briefly described, in each row there is be number of clusters¹⁸⁾ followed by cluster array. To make the simulation more realistic, two more features were included. Clusters are generated only with some probability (particles are detected only with this probability). And on the other hand noise clusters are produced too (they simulate background).

Analysis program reads the ROOT tree, finds the true clusters (hits produced by beam particles) and performs track fitting using formulae (2.5) and (2.6). Finally, the shift and slant calibration based on formulae (2.22) and (2.31) is done.

References

- [27] TOTEM-Technical Design Report-001, CERN-LHC-2004-002
- [28] Dr. Mario Deile and other people from TOTEM team, private communication

¹⁸⁾ Cluster in this meaning is piece of information describing a hit (i.e. detector number, charge distribution on strips, etc.)