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Computer Simulation of Physical Systems

Implementation of Space Charge Forces in BimBim

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1 Project Goals and scope

High intensity charged particle beams in circular accelerators are subject to various types of instability mechanisms leading to beam blow up and particle losses. The development of beam dynamics models for the understanding and mitigation of these instabilities plays a central role in the operation and design of state-of-the-art high intensity synchrotrons. One major aspects of the space charge forces is that they can be a limiting factor in the increase of the number of beam particles per bunch (referred to as the *intensity*) in low-energy machines such as the Proton Synchrotron (PS) or the Super Proton Synchrotron (SPS).

The project consists in extending a semi-analytical solver for self-consistent modes of oscillation taking into account the electromagnetic interactions of the particles in the beam with themselves, known as the *space charge* forces.

The solver makes use of the linearisation of the forces and a discretisation of the beam distribution in order to derive a matrix that can be analysed with a normal mode analysis. This is achieved by implementing the space charge forces in a preexisting simulation called BimBim. Two steps are defined, the first one consists in the derivation of an analytical expression for the space charge forces applied to the discretisation, while the second one consists in the implementation of those forces in BimBim. The results of the simulation are then benchmarked with analytical results. Once the simulation is proved reliable, it is used to investigate the influence of space charge forces on the planned SPS increase in intensity as part of the future upgrade of LHC into the High Luminosity LHC (HL-LHC) that will be taking place in 2025 and that will be able to increase by a factor of 10 the amount of collected collision data.

Space charge effects have already been extensively studied through both analytical and simulation approaches, but several reasons motivate the implementation of space charge effects in BimBim. Analytical approaches do not study the full oscillation spectrum of the beam, wich will be accessible through the semi-analytical approach of BimBim. Moreover, BimBim allows to simulate the combination of space charge effects with other effects affecting the beam stability. Finally, BimBim should also be able to handle any type of beam distribution, wich helps model realistic situations.

2 Physical Context

Let us first introduce the formalism employed in the description of the motion of particles along an accelerator. We will then present an overview of the main concepts related to the particle motion. The interaction of the beam with itself will then be studied through the so-called space charge mechanisms. The developments of this chapter are based on [1] and [2].

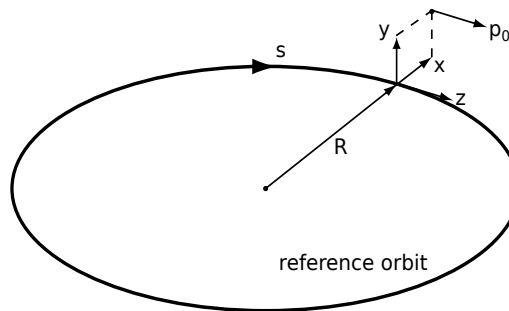


Figure 1: Frenet-Serret coordinate system

In order to introduce the formalism, let us consider the beam as a collection of identical charged particles, sent through a ring of radius R with momentum p_0 . The easiest way to handle the dynamics of the beam is to define a Frenet-Serret curvilinear coordinate system, as shown on Fig.1. This coordinate system relies on a *reference orbit*, i.e. the path of a *reference particle*. This path is parametrised with the path length s and is periodic with the ring circumference $C = 2\pi R$.

One can then define for a given curvilinear position s a local orthonormal coordinate system (x, y, z) moving along the particles. x and y denote the transverse position offset of the particle and z the longitudinal offset with respect to the reference particle.

The behaviour of a single particle is described by a pseudo 6 dimensional phase space $(x, x', y, y', z, \delta)$. *Pseudo-momenta* x' and y' are denoting the slope of the particle trajectory, i.e: $x' = dx/ds, y' = dy/ds$, and $\delta = (p - p_0)/p_0$ is its relative momentum offset.

The oscillations of the particles around the reference orbit will be influenced by different mechanisms introduced in the following sections. Nevertheless, we can already divide them in two distinct categories. *Single particle dynamics* comprises mechanisms such as particle focusing while *collective effects* concerns mechanisms such as the space charge forces.

2.1 Single Particle Dynamics

We will first investigate the effect of external fields on the single-particle dynamics. Considering a charged particle moving at a relativistic velocity $\vec{v}$ and carrying the charge q . The Lorentz force acting on the particle reads:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}). \quad (2.1)$$

An important consequence of this expression can already be stated: since the force due to the magnetic field is orthogonal to the particle motion, the electric field is the only component of the force able to accelerate the particle (and thus increase its kinetic energy), magnetic fields are thus exploited to deflect the particle.

When dealing with synchrotrons, one can make the reasonable assumption that the transverse velocities v_x, v_y are small in front of the longitudinal velocity v_z . This assumption is called the *paraxial assumption* and will allow us to consider that the only significant effect of the B fields on the particle motions will come from the terms involving the longitudinal speed v_z , i.e. that the transverse components of the magnetic fields can be neglected.

One can already deduce an important consequence of the paraxial approximation: the particles will follow a circular motion in both $x - z$ and $y - z$ planes:

$$\vec{F} = \vec{v} \times \vec{B} \approx q(-|v_z B_y| \vec{1}_x + |v_z B_x| \vec{1}_y). \quad (2.2)$$

This force is then balanced in both transverse directions by the centrifugal force:

$$F_c = m_p \gamma v_z^2 / \rho_x \vec{1}_x + m_p \gamma v_z^2 / \rho_y \vec{1}_y, \quad (2.3)$$

where m_p is the particle mass. Using the relativistic momentum of the particle $p = m_p \gamma v$ one can write this fundamental result:

$$|B_{y,x} \rho_{x,y}| = \left| \frac{\rho_{x,y}}{q} \right|. \quad (2.4)$$

linking the *beam rigidity* $B\rho$ to the particle momentum. In order to get rid of the dependence in the momentum p and the field B , we can expand the latter in a multipolar development and make use of the definition of magnet multipole strengths k_i . This leads to the following expression in the horizontal plane:

$$\frac{1}{\rho_x} = \frac{q B_y}{p_x} = k_1 - k_2 x + \dots \quad (2.5)$$

This expression shows clearly the purpose of the two main types of magnets used in accelerators. Let us analyse them in the horizontal plane. The dipolar magnet, with a vertical field $B_y = p/qk_1$, curves the reference orbit on a circular trajectory with constant radius. The quadrupolar magnet, with the fields $B_y = -p/qk_2 x$ and $B_x = p/qk_2 y$, bends the particles towards the reference orbit in one plane (the x horizontal one in our case) and away from the orbit in the other, with a bending radius inversely proportional to the transverse position of the particle.

2.2 Transverse betatron motion

We can now treat the case of a periodic *linear lattice*, i.e. a periodic sequence of dipole and quadrupole magnets along the ring. Starting from the expression of the magnetic force expressed earlier and expressing Newton's second law in the longitudinal s coordinate, and accounting only for the quadrupolar fields along the ring (providing the *focusing*) one can show that the transverse

motion of particles with the momentum $p_z = p_0$ is then governed by the following system of linear (and possibly coupled) second-order ordinary differential equations:

$$\frac{d^2}{ds^2} \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} \begin{pmatrix} K_x(s) \\ K_y(s) \end{pmatrix} \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = 0. \quad (2.6)$$

where $K_x(s) = k_{2y}(s)/p_z$ and $K_y(s) = k_{2x}(s)/p_z$ are functions of the path length s and periodic (at least) in the ring circumference C .

The solution of those equations is called the *betatron* motion and consists in a quasi-harmonic oscillation of the particles around the reference orbit. Its derivation and solving will not be explicitly done in the present report.

Let us consider a simple horizontal case. The motion equation reads as:

$$\frac{d^2}{ds^2} x(s) + K_x(s)x(s) = 0 \quad (2.7)$$

and is called the *Hill equation*. One can observe that this equation is the one of a pseudo-harmonic oscillator with a varying pulsation $\omega = \sqrt{K_x(s)}$. The solution is then given by:

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta(0)}{\beta(s)}} (\cos(\Psi(s)) + \alpha(0) \sin(\Psi(s))) & \sqrt{\beta(s)\beta(0)} \sin(\Psi(s)) \\ \frac{-(1+\alpha(0)\alpha(s)) \sin(\Psi(s)) + (\alpha(s)-\alpha(0)) \cos(\Psi(s))}{\sqrt{\beta(s)\beta(0)}} & \sqrt{\frac{\beta(0)}{\beta(s)}} (\cos(\Psi(s)) - \alpha(s) \sin(\Psi(s))) \end{pmatrix} \cdot \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix}, \quad (2.8)$$

where we introduced the *Twiss parameters* $\beta_x(s), \alpha_x(s) = -1/2\beta'_x(s)$ and $\Psi(s) = \int_0^s ds' \frac{1}{\beta(s')}$. The Twiss parameters are characteristic of the considered lattice, and their derivation from the lattice sequence $K_x(s)$ is usually done with the help of numerical solvers.

Rearranging the equations for x and x' , we can now derive an important motion invariant of the betatron oscillations called the *emittance* (or *Courant-Snyder invariant*)

$$\epsilon_x = \gamma_x x^2 + 2\alpha_x x x' + \beta_x'^2, \quad (2.9)$$

expliciting the fact that the particle is moving along the ellipsis of area $\sqrt{\pi}\epsilon_x$ parametrised by equations 2.8 in the transverse x -phase space.

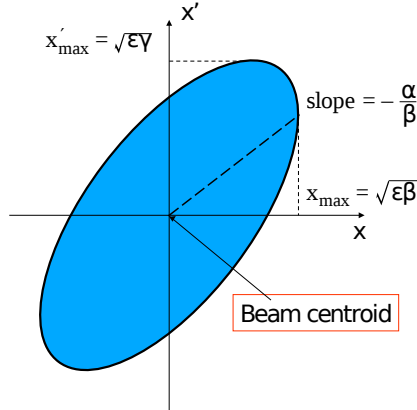


Figure 2: Transverse phase space: the particle evolves on an ellipsis

Eq.2.9 also indicates that the emittance is directly related to the amplitude of the oscillations, hence the beam size in the considered direction.

From Eq.2.8, we can derive the *one turn matrix* mapping the transverse particle coordinates from one turn to the next one.

$$M_{turn} = \begin{pmatrix} \cos(2\pi Q_x) + \alpha \sin(2\pi Q_x) & \beta \sin(2\pi Q_x) \\ -\gamma \sin(2\pi Q_x) & \cos(2\pi Q_x) - \alpha \sin(2\pi Q_x) \end{pmatrix}, \quad (2.10)$$

where $\gamma(s) = (1 + \alpha^2(s))/\beta_x(s)$ and Q_x is the *bare machine tune* given by

$$Q_x = \frac{1}{2\pi} \Psi(C) = \frac{1}{2\pi} \oint \frac{ds}{\beta_x(s)}, \quad (2.11)$$

denoting the pulsation of the particle in the transverse plane. This tune can be estimated if we consider the *smooth ring* approximation, i.e. if we consider constant optical functions along the ring circumference, as

$$Q_{x,y} \approx \frac{1}{2\pi} \frac{C}{\beta_{x,y}}. \quad (2.12)$$

One can show that this tune can be affected by a so-called *tune shift* if the quadrupolar field in the accelerator presents a gradient error, i.e: $K'(s) = K(s) + \Delta K(s)$. The tune shift is then equal to:

$$\Delta Q_x = \frac{1}{4\pi} \oint ds \Delta K(s) \beta_x(s). \quad (2.13)$$

Similarly to this transverse phase space rotation around a fixed point caused by the transverse focusing, the longitudinal focusing (achieved by oscillating radio-frequency cavities placed along the ring) causes a rotation of the single particle position in the longitudinal phase space (z, δ) with an angle equal to $2\pi Q_s$, where Q_s is the *synchrotron tune*. This movement is called the *synchrotron motion*.

2.3 Quadrupolar magnetic lenses

The case of a quadrupolar magnet of length l , with a vertical field equal to $B_y = -k_2 x$, reduces Eq.2.6 to the equation of a simple harmonic oscillator with $K = \frac{qk_2}{p_z}$. Its solution can be expressed through a *transfer matrix* linking the transverse position and momenta of a particle going in and out of the lens as:

$$\begin{pmatrix} x(l) \\ x'(l) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K}l) & \sin(\sqrt{K}l)/\sqrt{K} \\ -\sqrt{K}\sin(\sqrt{K}l) & \cos(\sqrt{K}l) \end{pmatrix} \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix}. \quad (2.14)$$

This transfer matrix can be simplified under the *thins lens* approximation, (i.e. if $\sqrt{K}l \ll 1$)

$$\begin{pmatrix} x(l) \\ x'(l) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix}, \quad (2.15)$$

which represents the fact that the quadrupole provide a "focusing kick" (defocusing if $f < 0$) to the particle, proportional to its initial transverse position, while leaving the latter unaffected. This is exactly the behavior of an optical thin lens of focal length $f = 1/KL$, justifying the name of *magnetic lens*.

2.4 Collective effects: Direct Space Charge

We will now approach the influence of the self field of the beam on its dynamics. This mechanism is caused by the beam as a whole and cannot be handled with the single particle approach that we derived previously. In fact, the individual oscillations of a single particle (generally referred as *incoherent* motion) cannot even be measured in the accelerators, as only averaged quantities are accessible with dedicated diagnostics.

The approach used to deal with those collective effects will require to assume a certain particle distribution in the different degrees of freedom, randomly distributed over the beam particles. This will allow us to study its *coherent* motion, i.e: the evolution of its statistical moments, which can be measured by instruments installed along the accelerator. If we consider a distribution function $f(x, y, z, x', y', \delta; s)$ normalized over the 6D phase space, the first order moment (referred to as the *beam centroid*) will read as:

$$\langle u \rangle(s) = \int d^6 \xi u f(\xi; s), \quad (2.16)$$

where $\xi = (x, y, z, x', y', \delta)$ is a vector of the phase space and the integration being done on the entire space. The n-th order statistical moment of the beam is thus given by:

$$u_n(s) = \int d^6 \xi (u - \langle u \rangle)^n f(\xi; s). \quad (2.17)$$

From this definition follows that the second order moment of the beam, or its *root mean square* size is defined as

$$\sigma_x^2(s) = \int d^6\xi (x - \langle x \rangle)^2 f(\xi; s), \quad (2.18)$$

with a similar expression in the y, z and δ dimensions.

We will now summarise some important results about direct space charge dynamics.

2.5 Direct Space charge dynamics

Following the developments from [2], let us investigate the space charge effect with a single bunch of N_b particles (N_b is referred to as the bunch *intensity*) distributed according to a full 6 dimensional Gaussian distribution, which will be expressed in the longitudinal phase space as:

$$\Psi_l(z, \delta) = \frac{N_b}{2\pi\sigma_z\sigma_\delta} e^{-\frac{z^2}{2\sigma_z^2}} e^{-\frac{\delta^2}{2\sigma_\delta^2}} = \lambda(z) \frac{1}{\sqrt{2\pi\sigma_\delta}} e^{-\frac{\delta^2}{2\sigma_\delta^2}}. \quad (2.19)$$

And in the transverse plane as:

$$\rho(x, y; z) = \frac{\lambda(z)}{2\pi\sigma_x\sigma_y} e^{-\frac{x^2}{2\sigma_x^2}} e^{-\frac{y^2}{2\sigma_y^2}}, \quad (2.20)$$

where $\lambda(z)$ is the beam *line density*.

The paraxial nature of the beam allows us to make useful assumptions: in the beam rest frame particles move at very slow relative velocities, which implies that B can be neglected and that we can consider the situation as an electrostatic problem. Because of relativistic effects, we can also consider the longitudinal component of the electric field to be negligible.

Under those assumptions, one can show that the equation of transverse motion for the beam including the space charge effects are given by:

$$\frac{d^2}{ds^2} \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} + \begin{pmatrix} K_x(s) \\ K_y(s) \end{pmatrix} \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = \frac{q}{\beta\gamma p_0 c} \begin{pmatrix} E_x(s) \\ E_y(s) \end{pmatrix}. \quad (2.21)$$

The transverse electric field generated (in the laboratory frame) by the beam is derived in [3] as:

$$E_u(u - \langle u \rangle) = \frac{\lambda}{4\pi\epsilon_0} u \int_0^\infty dt \frac{\exp(-\frac{x^2}{2\sigma_x^2+t} - \frac{y^2}{2\sigma_y^2+t})}{(\sigma_u^2 + t) \sqrt{(\sigma_x^2 + t)(\sigma_y^2 + t)}}; u = x, y, \quad (2.22)$$

which can be linearised around the section centroid $\langle u \rangle$ as:

$$\frac{\lambda}{2\pi\epsilon_0} \frac{u - \langle u \rangle}{\sigma_u[\sigma_x + \sigma_y]}. \quad (2.23)$$

Defining the *space charge perveance* as:

$$K^{SC}(z) = \frac{q\lambda(z)}{2\pi\epsilon_0\beta\gamma^2 p_0 c}. \quad (2.24)$$

The single particle equation of motion becomes for the horizontal plane:

$$\frac{d^2 x}{ds^2} + K_x(s)x = \frac{K^{SC}(z)x}{\sigma_x[\sigma_x + \sigma_y]}. \quad (2.25)$$

and conversely for the vertical plane. Taking the expectation value over both sides of the equality, we obtain an equation for the motion of the *section centroid* $\langle x \rangle(z)$:

$$\frac{d^2}{ds^2} \langle x \rangle(z) + K_x(s) \langle x \rangle(z) = 0, \quad (2.26)$$

which shows the absence of space charge effects on the motion of the section centroid. Combining the two previous equations leads to

$$\frac{d^2}{ds^2} (x - \langle x \rangle) + (K_x(s) - \frac{K^{SC}(z)x}{\sigma_x[\sigma_x + \sigma_y]}) (x - \langle x \rangle) = 0. \quad (2.27)$$

Where we see that the focusing $K_x(s)$ is depressed by the space charge term. This carries a very important physical meaning: the repulsive space charges act exactly as a defocusing quadrupolar lens distributed along the entire ring and centered around the section centroid. This particular aspect will be the basis of the approach of the simulation.

It is also worth noting that this expression indicates that the space charge effect increases with the line density of the bunch $\lambda(z)$, and decreases with its energy and transverse sizes. This additional defocusing term can be considered as a quadrupolar error distributed along the ring, allowing us to deduce the tune shift induced by the direct space charges:

$$\Delta Q_x^{SC}(z) = \frac{1}{4\pi} \oint ds \Delta K \beta_x(s) = \frac{K^{SC}(z)}{4\pi} \oint ds \frac{\beta_x(s)}{\sigma_x(s)[\sigma_x(s) + \sigma_y(s)]}. \quad (2.28)$$

or expressing it more explicitly:

$$\Delta Q_x^{SC}(z) = -\frac{q\lambda(z)}{8\pi\epsilon_0 m_p c^2 \beta^2 \gamma^3} \oint ds \frac{\beta_x(s)}{\sigma_x(s)[\sigma_x(s) + \sigma_y(s)]}. \quad (2.29)$$

which reduces in the smooth ring approximation to:

$$\Delta Q_x^{SC}(z) = \frac{q\lambda(z)}{8\pi\epsilon_0 m_p c^2 \beta^2 \gamma^3} \frac{\beta_x C}{\sigma_x[\sigma_x + \sigma_y]}. \quad (2.30)$$

3 Circulant Matrix Model

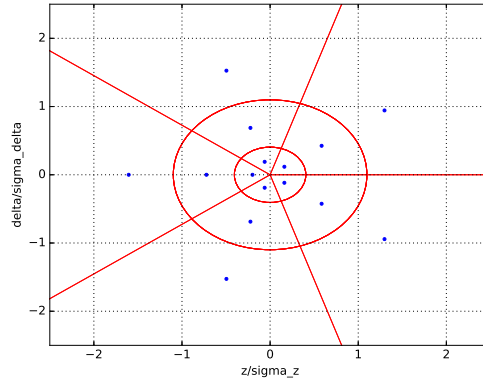


Figure 3: Longitudinal phase space decomposition in 5 slices and 3 rings.

We will now present the physical model of the bunch, based on the developments made in [1].

As shown on Fig.3, the Circulant Matrix Model (CMM) relies on a polar decomposition of the longitudinal phase space distribution $\Psi(R)$ of a particle bunch, in the normalized phase space $R = \sqrt{(s/\sigma_s)^2 + (\delta/\sigma_\delta)^2}$, where σ_s and σ_δ are the bunch second order moments defined in the previous section (they are respectively referred to as the *r.m.s bunch length* and its *momentum deviation*). The phase space is decomposed into $N_r - 1$ radial boundaries and N_s angular boundaries, defining a decomposition of $N_s \cdot N_r$ cells. The radial boundaries $r_{b,i}$ can be chosen arbitrarily (as long as $r_{b,0} = 0$), while the angular boundaries are given by $\theta_{b,i} = 2\pi i/N_s, i = 0, \dots, N_s - 1$. Each cells get attributed a center, whose coordinates are defined as follows:

- $r_i = \int_{r_{b,i-1}}^{r_{b,i}} R dR \Psi(R);$
- $\theta_j = \frac{\theta_{b,j} + \theta_{b,j+1}}{2} = \frac{\pi(2j+1)}{N_s}.$

Considering the cell (i, j) , the cartesian coordinates of its center in the non normalized space are given by

- $z_{i,j} = r_i \cos(\theta_j) \sigma_z$;
- $\delta_{i,j} = r_i \sin(\theta_j) \sigma_\delta$.

Every cell is also given a certain weight accounting for the proportion of the bunch it contains, computed as follows:

$$w_{i,j} = \int_{\theta_{b,j}}^{\theta_{b,j+1}} d\theta \int_{r_{b,i-1}}^{r_{b,i}} R dR \Psi(R) = \frac{2\pi}{N_s} \int_{r_{b,i-1}}^{r_{b,i}} R dR \Psi(R) . \quad (3.1)$$

In the case of a Gaussian distribution, we can choose the radial boundaries such that all weights equal $1/(N_r N_s)$. This is achieved with the following expression:

$$e^{-r_i} - e^{-r_{i+1}} = 1/N_r . \quad (3.2)$$

For the following developments, we will make the assumption that the transverse distribution of every cell (i, j) is given by a 2D-Gaussian of mean $(x_{i,j}, x'_{i,j})$ and variances $\sigma_x, \sigma_{x'}$. We will not consider the y transverse plane, but the results can be easily extended to include it.

We can now derive the one turn transfer matrix M (of size $2 \cdot N_r N_s$) mapping the mean transverse position of each cells center from one turn to the other. It is defined formally as follows :

$$\begin{pmatrix} x_{1,k+1} \\ x'_{1,k+1} \\ x_{2,k+1} \\ x'_{2,k+1} \\ \vdots \\ x_{N_r N_s, k+1} \\ x'_{N_r N_s, k+1} \end{pmatrix} = M \cdot \begin{pmatrix} x_{1,k} \\ x'_{1,k} \\ x_{2,k} \\ x'_{2,k} \\ \vdots \\ x_{N_r N_s, k} \\ x'_{N_r N_s, k} \end{pmatrix} \quad (3.3)$$

where $x_{n,k}$ denotes the mean transverse coordinate at turn k for the cell (i, j) with $n = N_s(i-1) + j$, and conversely for $x'_{n,k}$. It is worth noting that the one turn matrix is usually defined for any given number of bunches and can also be defined for the two counter-circulating beams of a collider. We will nevertheless limit the presentation to the basic case of one transverse dimension of a single bunch in a single beam.

This matrix needs to take into account any effect that we want to model in the transport of the beam. We will focus on the synchro-betatron transport and space charge effects.

3.1 Synchro-betatron transport

As mentioned in Sec.2.2, the synchrotron motion over a whole turn corresponds to a rotation in the normalized longitudinal phase space of angle $2\pi Q_s$. Its transfer matrix can hence be written as:

$$S_r = P_{N_s}^{N_s Q_s} , \quad (3.4)$$

where P_{N_s} is the following circulant permutation matrix of size N_s causing a rotation of angle $1/N_s$:

$$P_{N_s} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ & 0 & 1 & \\ & & \ddots & \ddots & \vdots \\ & & & 0 & 1 \\ 1 & \dots & 0 & 0 \end{pmatrix} . \quad (3.5)$$

Accounting for the fact that each cell will stay on the same ring, the synchrotron motion can therefore be modeled by:

$$S_0 = I_{N_r} \otimes S_r . \quad (3.6)$$

Recalling that the betatron motion can also be expressed as a rotation angle $2\pi Q_x$ on the surface of the transverse phase space ellipsis, each cell will be affected by the following rotation:

$$B_0 = \begin{pmatrix} \cos(2\pi Q_x) & \beta_x \sin(2\pi Q_x) \\ -\frac{1}{\beta_x} \sin(2\pi Q_x) & \cos(2\pi Q_x) \end{pmatrix}. \quad (3.7)$$

where we cleverly chose the starting point such that $\alpha(0) = 0$. The one turn matrix is given by the composition of betatron and synchrotron rotations for each cell, which can be simply written as:

$$M_0 = S_0 \otimes B_0. \quad (3.8)$$

This matrix is called the *unperturbed synchro-betatron one turn matrix* for a single bunch.

3.2 Direct Space Charge matrix

Here lies the first objective of the project: finding an analytical expression of a matrix modeling the space charge effects presented in Sec.2.5.

Recalling that the effect of the field generated by the beam Gaussian distribution on a single particle can be re-expressed as the effect of a quadrupolar lens distributed along the whole ring with strength $\frac{K^{SC}C}{\sigma_x(\sigma_x + \sigma_y)}$ and centered around the section centroid $\langle x \rangle(z)$. We can write this defocusing effect over one full turn on a single particle with the help of the transfer matrix of a quadrupolar lens in the thin approximation (Eq.2.15):

$$\begin{cases} x_{k+1}(z) = x_k(z) \\ x'_{k+1}(z) = x'_k(z) + \frac{K^{SC}(z)C}{\sigma_x(\sigma_x + \sigma_y)}(x_k(z) - \langle x \rangle(z)) \end{cases} \quad (3.9)$$

Those are the coordinates transform for a single particle over one turn. In order to derive the space charge matrix for the CMM, we need to derive the kick received by an entire cell of the decomposition. Let us compute this kick for a single cell with centroid x , located at longitudinal location z . Recalling the assumption that every cell was distributed as a Gaussian $\Psi(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \exp(-x^2/2\sigma_x^2)$, the integrated kick, or *coherent* kick, can be expressed as the following integral:

$$\begin{aligned} \Delta x'_{coh}(x; z) &= \int_{-\infty}^{\infty} dX \Delta x'(X) \Psi(X - x) \\ &= \frac{\Delta x'(x; z)}{2}; \quad x \ll \sigma_x, \sigma_y \\ &= \frac{K^{SC}(z)C}{2\sigma_x(\sigma_x + \sigma_y)}(x_k(z) - \langle x \rangle(z)) \\ &= \alpha^{sc}(z)(x_k(z) - \langle x \rangle(z)). \end{aligned} \quad (3.10)$$

where we had to make use of the non-linear single particle kick, derived in [4], and where the condition $x \ll \sigma_x, \sigma_y$ is met since we are interested in oscillations of small amplitude, corresponding to the initiation of an instability.

The space charge matrix can be defined similarly to the unperturbed one turn matrix:

$$\begin{pmatrix} x_{1,k+1} \\ x'_{1,k+1} \\ x_{2,k+1} \\ x'_{2,k+1} \\ \vdots \\ x_{N_r N_s, k+1} \\ x'_{N_r N_s, k+1} \end{pmatrix} = M_{SC} \cdot \begin{pmatrix} x_{1,k} \\ x'_{1,k} \\ x_{2,k} \\ x'_{2,k} \\ \vdots \\ x_{N_r N_s, k} \\ x'_{N_r N_s, k} \end{pmatrix} \quad (3.11)$$

so that the full turn matrix will be given by the product of the unperturbed one turn matrix and the space charge matrix:

$$M = M_0 \cdot M_{SC}. \quad (3.12)$$

The last aspect to be solved is the calculation of the section centroid $\langle x \rangle(z)$. This calculation is not trivial, since we need a mean of exploiting the polar decomposition of the CMM to express the value of the section centroid at the Cartesian coordinate z .

Recalling the definition of the Gaussian distribution of Eq.2.19, the section centroid is expressed in the continuous case as

$$\langle x \rangle(z) = \frac{\int_{-\infty}^{\infty} d\delta \, x(z, \delta) \Psi(z, \delta)}{\int_{-\infty}^{\infty} \Psi(z, \delta)} = \int_{-\infty}^{\infty} d\delta \, x(z, \delta) \frac{e^{-\delta^2/2\sigma_\delta^2}}{\sqrt{2\pi}\sigma_\delta}. \quad (3.13)$$

In order to apply this expression to our phase space decomposition, one needs to find an estimate expression of $\langle x \rangle(z)$.

The approach consists in considering $x(z, \delta)$ to be constant over every cells and to be equal to the cell centroid x_i , i.e. the mean value of x over the cell. This allows us to write the integral as a sum that can be evaluated analytically, as long as we know every intersection of the line $z = z_i$ with the cells boundaries, called δ_j , ($j = 1, \dots, n$).

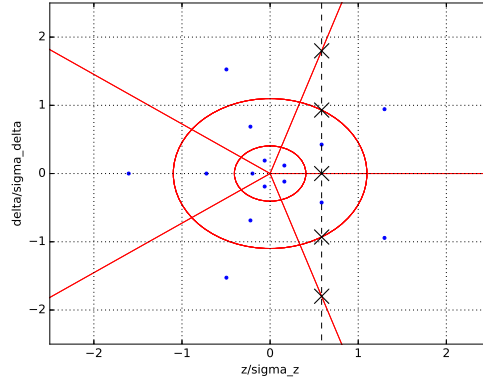


Figure 4: Longitudinal phase space decomposition in 5 slices and 3 rings: the 5 intersections of the line $z = z_6$ with the decomposition are marked by the crosses.

The integral reads now as:

$$\langle x \rangle(z) \approx x_0 \int_{-\infty}^{\delta_1} d\delta \frac{e^{-\delta^2/2\sigma_\delta^2}}{\sqrt{2\pi}\sigma_\delta} + \sum_{i=1}^{n-1} x_i \int_{\delta_i}^{\delta_{i+1}} d\delta \frac{e^{-\delta^2/2\sigma_\delta^2}}{\sqrt{2\pi}\sigma_\delta} + x_n \int_{\delta_n}^{\infty} d\delta \frac{e^{-\delta^2/2\sigma_\delta^2}}{\sqrt{2\pi}\sigma_\delta}. \quad (3.14)$$

We see three types of integration domains appearing in the finite sum: from $]-\infty, \delta_1]$, $[\delta_i, \delta_{i+1}]$ and $[\delta_n, \infty[$, a computable solution can be stated for each domain as follows:

$$\begin{aligned} \int_{-\infty}^{\delta_1} d\delta \frac{e^{-\delta^2/2\sigma_\delta^2}}{\sqrt{2\pi}\sigma_\delta} &= \frac{\text{erf}\left(\frac{\delta_1}{\sqrt{2}\sigma_\delta}\right) + 1}{2} \\ \int_{\delta_i}^{\delta_{i+1}} d\delta \frac{e^{-\delta^2/2\sigma_\delta^2}}{\sqrt{2\pi}\sigma_\delta} &= \frac{\text{erf}\left(\frac{\delta_{i+1}}{\sqrt{2}\sigma_\delta}\right) - \text{erf}\left(\frac{\delta_i}{\sqrt{2}\sigma_\delta}\right)}{2} \\ \int_{\delta_n}^{\infty} d\delta \frac{e^{-\delta^2/2\sigma_\delta^2}}{\sqrt{2\pi}\sigma_\delta} &= \frac{1 - \text{erf}\left(\frac{\delta_n}{\sqrt{2}\sigma_\delta}\right)}{2}. \end{aligned} \quad (3.15)$$

Where the *error function* is $\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t \exp(-t^2) dt$.

We can now rewrite Eq.3.14 by defining the linear form $\langle X \rangle(z)$ acting on the space phase decomposition so that:

$$\langle x \rangle(z) \approx \langle X \rangle(z) \cdot (x_{1,k} \quad x'_{1,k} \quad \dots \quad x_{N_r N_s, k} \quad x'_{N_r N_s, k})^T. \quad (3.16)$$

This linear form allows us to write the one turn kick provided by space charge effect of any cell j in function of the phase space coordinates of the previous turn:

$$\begin{aligned} x'_{j,k+1} &= x'_{j,k} + \alpha^{sc}(z)(x_{j,k} - \langle x \rangle(z)) \\ &\approx x'_{j,k} + \alpha^{sc}(z)x_{j,k} - \alpha^{sc}(z)\langle X \rangle(z) \cdot (x_{1,k} \quad x'_{1,k} \quad \dots \quad x_{N_r N_s, k} \quad x'_{N_r N_s, k})^T \end{aligned} \quad (3.17)$$

The complete one turn coordinates transformations over one turn for cell j is then given by:

$$\begin{cases} x_{j,k+1} = x_{j,k} \\ x'_{j,k+1} \approx x'_{j,k} + \alpha^{sc}(z)x_{j,k} - \alpha^{sc}(z)\langle X \rangle(z) \cdot (x_{1,k} \quad x'_{1,k} \quad \dots \quad x_{N_r N_s, k} \quad x'_{N_r N_s, k})^T \end{cases} \quad (3.18)$$

We derived now all the tools needed to systematically build the space charge matrix. It is now clear that space charge effects introduce a coupling between the different cells along the δ axis, this was expected from the physics of the problem, since the space charge effect we modeled acts only on the transverse dimension and has no longitudinal component.

For the clarity of the explanation, we present hereafter the complete matrix derivation in the simple case of a basis containing 4 slices and 1 ring presented on Fig.5.

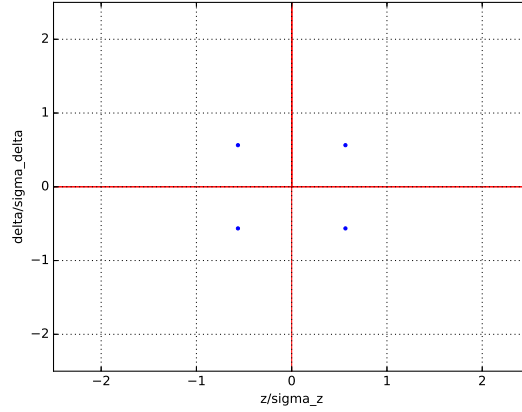


Figure 5: Phase space decomposition in 4 slices and 1 ring.

Following the space charge kick derived in equation 3.10. We obtain the following linear coupled system of equations:

$$\begin{cases} x_{1,k+1} = x_{1,k} \\ x'_{1,k+1} = x'_{1,k} + \alpha(z_1)(x_{1,k} - \langle x \rangle(z_1)) \\ x_{2,k+1} = x_{2,k} \\ x'_{2,k+1} = x'_{2,k} + \alpha(z_2)(x_{2,k} - \langle x \rangle(z_2)) \\ x_{3,k+1} = x_{3,k} \\ x'_{3,k+1} = x'_{3,k} + \alpha(z_3)(x_{3,k} - \langle x \rangle(z_3)) \\ x_{4,k+1} = x_{4,k} \\ x'_{4,k+1} = x'_{4,k} + \alpha(z_4)(x_{4,k} - \langle x \rangle(z_4)) \end{cases} \quad (3.19)$$

Making use of the estimation of $\langle x \rangle(z)$ derived in Eq.3.14. The system becomes:

$$\begin{cases}
x_{1,k+1} = x_{1,k} \\
x'_{1,k+1} = x'_{1,k} + \alpha(z_1)(x_{1,k} - x_{1,k} \int_0^\infty d\delta x(z_1, \delta) \Psi(\delta) - x_{4,k} \int_{-\infty}^0 d\delta x(z_1, \delta) \Psi(\delta)) \\
x_{2,k+1} = x_{2,k} \\
x'_{2,k+1} = x'_{2,k} + \alpha(z_2)(x_{2,k} - x_{2,k} \int_0^\infty d\delta x(z_2, \delta) \Psi(\delta) - x_{3,k} \int_{-\infty}^0 d\delta x(z_2, \delta) \Psi(\delta)) \\
x_{3,k+1} = x_{3,k} \\
x'_{3,k+1} = x'_{3,k} + \alpha(z_3)(x_{3,k} - x_{2,k} \int_0^\infty d\delta x(z_1, \delta) \Psi(\delta) - x_{3,k} \int_{-\infty}^0 d\delta x(z_3, \delta) \Psi(\delta)) \\
x_{4,k+1} = x_{4,k} \\
x'_{4,k+1} = x'_{4,k} + \alpha(z_4)(x_{4,k} - x_{1,k} \int_0^\infty d\delta x(z_4, \delta) \Psi(\delta) - x_{4,k} \int_{-\infty}^0 d\delta x(z_4, \delta) \Psi(\delta))
\end{cases} \quad (3.20)$$

Where $\Psi(\delta) = \frac{e^{-\delta^2/2\sigma_\delta^2}}{\sqrt{2\pi}\sigma_\delta}$. Let us rewrite the system in a matricial form to explicit the expression of the space charge matrix:

$$\begin{pmatrix} x_{1,k+1} \\ x'_{1,k+1} \\ x_{2,k+1} \\ x'_{2,k+1} \\ x_{3,k+1} \\ x'_{3,k+1} \\ x_{4,k+1} \\ x'_{4,k+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha(z_1)[1 - c^+(z_1)] & 1 & 0 & 0 & 0 & 0 & -\alpha(z_1)c^-(z_1) & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha(z_2)[1 - c^+(z_2)] & 1 & -\alpha(z_2)c^-(z_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\alpha(z_3)c^-(z_3) & 0 & \alpha(z_3)[1 - c^+(z_3)] & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \alpha(z_1)c^-(z_1) & 0 & 0 & 0 & 0 & 0 & 0 & \alpha(z_4)[1 - c^+(z_4)] \end{pmatrix} \begin{pmatrix} x_{1,k} \\ x'_{1,k} \\ x_{2,k} \\ x'_{2,k} \\ x_{3,k} \\ x'_{3,k} \\ x_{4,k} \\ x'_{4,k} \end{pmatrix} \quad (3.21)$$

where we defined:

$$\begin{cases} c^+(z) = \int_0^\infty d\delta x(z, \delta) \Psi(\delta) \\ c^-(z) = \int_{-\infty}^0 d\delta x(z, \delta) \Psi(\delta) \end{cases} \quad (3.22)$$

The algorithm developed to systematically derive this space charge matrix is presented in Sec.4.1. Nevertheless, this explicit derivation of a specific case helps grasp the subtleties of the problem. It also explicitly shows that in the absence of space charge forces (i.e. if $\alpha^{sc}(z) = 0$), the space charge matrix reduces to the identity matrix. This fact will be exploited in order to ensure the proper functioning of the simulation.

3.3 Normal mode analysis

One of the principal use of the CMM is that the stability of the beam can be easily assessed from the analysis of the eigenvalues of the one turn matrix. Let W be the change of basis matrix diagonalizing M into the diagonal matrix D carrying the eigenvalues of M on its diagonal:

$$M = W^{-1}DW \quad (3.23)$$

The behaviour of the beam after many turns is described by the n-turn matrix:

$$M^n = W^{-1}D^nW \quad (3.24)$$

where D^n carries the n-th power of the eigenvalues on its diagonal. Let us decompose the initial configuration of the system in the basis of the eigenvectors $v(n=0) = \sum_i a_i V_i$, where V_i is the eigenvector corresponding to the eigenvalue λ_i . After n turns, the system will be in the configuration:

$$\sum_j a_j \lambda_j^n V_j = \sum_j a_j e^{-in2\pi Q_j} V_j, Q_i = i \log(\lambda_i)/(2\pi) \quad (3.25)$$

The consequence of this expression is quite straightforward: if the imaginary part of Q_j is strictly positive, any system with a non-zero component of this mode ($a_j \neq 0$) will eventually diverge and

be called *unstable*. Conversely, any mode associated with an eigenvalue with negative real value will be spontaneously damped.

The eigenvalues and eigenmodes of the unperturbed synchro-betatron motion can be analytically derived. Assuming N_s to be odd for simplicity, the unperturbed tune spectrum is given by :

$$Q_0 = \left\{ Q_{coh} = \pm(Q_x + n_a Q_s); n_a \in \mathbb{Z}, |n_a| < \frac{N_s - 1}{2} \right\}, \quad (3.26)$$

where Q_x and Q_s are here the fractional parts of the betatron and synchrotron tunes. Eq.3.26 indicates that the eigenvalues are centered around Q_x and spaced by a distance of Q_s . The corresponding eigenvectors are presented on Fig.6, their number of nodes is equal to the *azimuthal* mode number n_a .

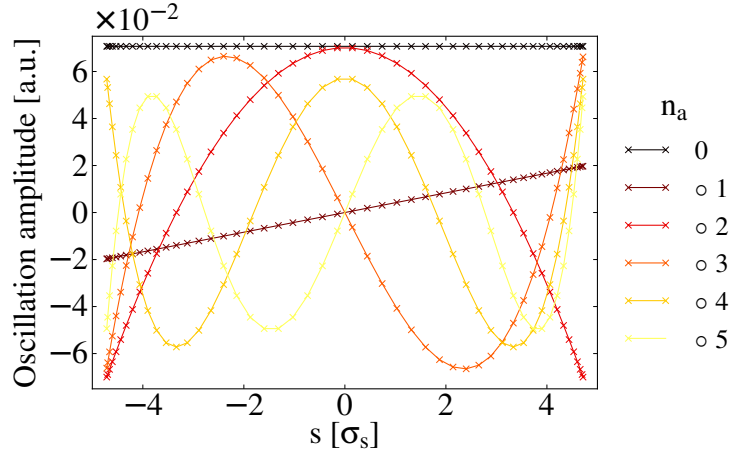


Figure 6: Azimuthal modes of the unperturbed synchro-betatron motion, taken from [1]

It is worth noting that each eigenvalue is degenerated N_r times, those degenerated modes with identical azimuthal number are called the *radial* modes. This degeneration is quickly lifted when considering the actual perturbed one turn matrix.

Another important observation has to be stated about the eigenvalues of the unperturbed one-turn matrix: its spectrum is composed by two symmetric positive and negative parts, representing the fact that the particles rotating in the opposite direction are also valid physical solutions of the synchro-betatron motion.

4 BimBim

Developped by Xavier Buffat as part of his thesis [1], BimBim is an implementation of the CMM in Python. It aims at systematically building the one turn matrix for both counter-circulating beams of a collider. The modelisation of several phenomenons is already implemented. Those phenomena include the interactions of both beams with each other (the *beam-beam* forces), the interaction of the beams with the pipe surrounding it (*coupling impedance* forces) and transverse feedback. The code takes advantage of the object-oriented possibilities of Python and make use of sparse matrices for memory considerations.

As mentioned previously, the aim of this project is to extend BimBim by implementing a module accounting for the direct space charge effects. For simplicity of development, we only consider the horizontal transverse dimension of one beam composed of a single bunch. This should obviously be extended to any beam configuration and to both dimensions of the transverse plane in future work.

The developments in the following sections will only be done accounting for this simplified beam configuration but can easily be extended to the general case.

BimBim is used to generate the one turn matrix, taking as input the parameters listed in Tab.1. Those parameters include the machine optics parameters and the beam parameters in the smooth ring approximation. A second input must include a so-called *action sequence*, mentioning any effect that should be simulated.

Proton energy	Q'	N_b	$\epsilon_{x,y}$	C	σ_z	σ_δ	Q_s	$Q_{x,y}$	N_r	N_s
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Table 1: BimBim input parameters

The eigenvalues of the one turn matrix are then computed by diagonalization. This last step can be a computational bottleneck in the execution time of the code. The real part of those eigenvalues is converted into tune shifts accordingly to Eq.3.25.

4.1 Space charge matrix implementation

Recalling the derivation of the space charge matrix done in Sec.2.5, the remaining objective was to implement an efficient way of systematically building the matrix for any given number of slices and rings. The general procedure for generating the space charge matrix can be presented as follows:

For every cell centered around (z_i, δ_i) ($i = 1 \dots, N_r N_s$):

- **Step 1:** find the intersections between the cell boundaries and the vertical line $z = z_i$, remove the duplicates, and order them in a list of increasing order;
- **Step 2:** compute the integrals for the three types of integration domains accordingly to Eq.3.15;
- **Step 3:** return the matrix line $\langle X \rangle(z)$;
- **Step 4:** fill in accordingly with Eq.3.14 the two lines of the matrix corresponding to $x_{i,k+1}$ and $x'_{i,k+1}$.

Step 1: the implementation of this step is quite straightforward. The intersections can be of two nature: either with the radial boundaries or with the angular ones. Some computation can be saved by noticing that the angular intersections only exists if z_i has the same sign that $\cos(\theta_{b,j})$ and that the radial intersections only exist if $z_i^2 + \delta_i^2 \leq r_{b,j}^2$

Step 3: in order to return the vector line $\langle X \rangle(z_i)$, the integrals computed in step 2 must be filled in a vector line according to the cell they correspond to, i.e, we must retrieve what cell is contained between two intersections δ_j and δ_{j+1} , with longitudinal coordinates z_i . This cell identification was implemented by checking for which cell (m, n) the following inequalities are simultaneously respected:

$$\begin{cases} r_{m,b} \leq r_j \leq r_{m+1,b} \\ r_{m,b} \leq r_{j+1} \leq r_{m+1,b}; m = 0, \dots, N_r - 1 \end{cases} \quad \begin{cases} \theta_{n,b} \leq \theta_j \leq \theta_{n+1,b} \\ \theta_{n,b} \leq \theta_{j+1} \leq \theta_{n+1,b}; n = 0, \dots, N_s - 1 \end{cases} \quad (4.1)$$

where

$$\begin{cases} r_j = \sqrt{z_i^2 + \delta_j^2} \\ \theta_j = \arctan\left(\frac{\delta_j}{z_i}\right) \end{cases} \quad (4.2)$$

The index of the retrieved cell is then given by $I = N_s(n - 1) + m$

This step is the most computationally costly of the algorithm since it needs to execute a number of checks equal to the number of cells in the basis, for every couple of intersections, and this again for every cell in the basis since $\langle X \rangle(z_i)$ needs to be computed for every cell. An improvement of this cell retrieving method could improve the running time of the code.

One can notice that computation of $\langle X \rangle(z_i)$ for each cell depends only on the basis parameters N_r and N_s and does not depend in any way of the beam or machine parameters. This calculation can thus be executed only once for a certain basis and then kept in memory, saving a good amount of calculations. The part of the simulation requiring the most computations is thus still the diagonalization of the one turn matrix. Nevertheless, the total running time of the simulation is not constraining in most cases.

5 Benchmarks

After having thoroughly compared the space charge matrices generated by BimBim with explicit computation of the space charge matrix for low numbers of cells, the method implemented can be benchmarked against theoretically predicted results.

Analytical results for the tune shift caused by space charge effects on the azimuthal modes have been derived in [5] in the case of a so-called *airbag distribution* in the longitudinal plane, for which it is assumed that the particles have a unique action for over a uniform distribution of angles. We chose to try reproducing those results with our BimBim module accounting for synchro-betatron motion and space charge forces because in the case of a single ring, one can show that the Gaussian CMM longitudinal distribution is almost equivalent to an airbag distribution.

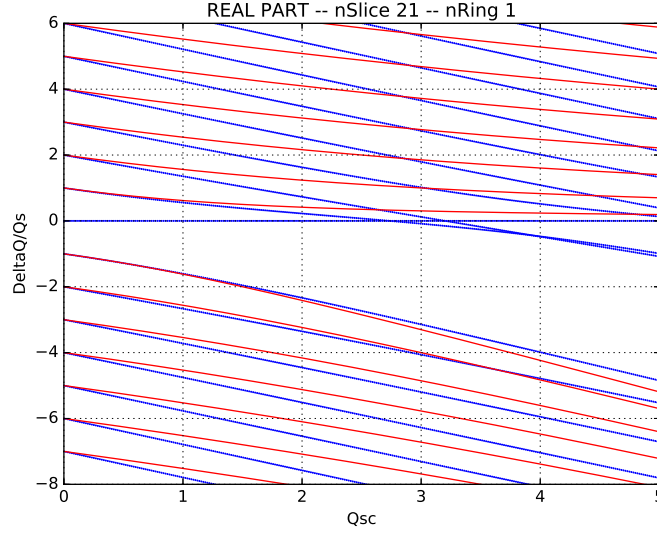


Figure 7: BimBim simulated tune shifts (blue) versus analytical results (red)

Fig.7 presents the comparison between the theoretical results of [5] and the BimBim output in function of the *space charge parameter* $Q_{sc} = \Delta Q_{sc}(z=0)/Q_s$, where $\Delta Q_{sc}(z)$ is given by Eq.2.30. The analytical spectrum is given by the following expression:

$$\begin{aligned} \frac{\Delta Q_{n_a}}{Q_s} &= \frac{Q_{n_a} - Q_x}{Q_s} \\ &= -\frac{Q_{sc}}{2} \pm \sqrt{\frac{Q_{sc}^2}{4} + (n_a)^2} \end{aligned} \quad (5.1)$$

where the $+$ is used for $n_a > 0$ and conversely for the $-$. The vertical axis is plotted in function of $\frac{\Delta Q_{n_a} - Q_x}{Q_s}$

Several observations can be stated:

- the simulated tune shifts join the unperturbed tune shifts for $Q_{sc} = 0$, this confirms the expected behavior of the space charge matrix at the origin: it becomes the identity matrix;
- just as predicted theoretically, mode 0 stays unperturbed by simulated space charge effects;
- modes -1 , 0 and 1 seem to be well reproduced by the simulation for small values of Q_{sc} ;
- the crossing of mode 0 by positive modes is not predicted by theoretical results, the argument about the physical reality of those crossings is still subject to discussion, but they might indicate an upper bound to the limit of the validity region of the simulation.

In conclusion, this benchmark cannot allow us to confirm the validity of the method we implemented, nor does it infirm the results below the crossing of mode 0 by mode 1.

This fact led us to reconsider a certain point of the method that was implemented: indeed, the use of the *thin lens* approximation in the derivation of the space charge kick is easily questionable. For the thin lens approximation to be valid, one should make sure that the following condition is respected:

$$\sqrt{\frac{K^{SC}(z)}{2\sigma_x[\sigma_x + \sigma_y]}} C \ll 1 \quad (5.2)$$

this condition is in fact not respected in any realistic machine configuration.

This issue led to a slightly modified implementation: instead of building the one turn matrix composed of the product of a one turn transport matrix with a space charge matrix, we construct it by multiplying N_{chunk} so-called *chunks* composed of the product of a transport matrix and a space charge matrix, only accounting for the transport of the beam of a distance of C/N_{chunk} . This allows the thin lens approximation to be respected for every space charge matrix. As shown on Fig.8, this modified method didn't reduce the discrepancies between the analytical values and the simulation results. This allows us to conclude that those are not caused by the thin lens assumption and that their exact cause is still unknown.

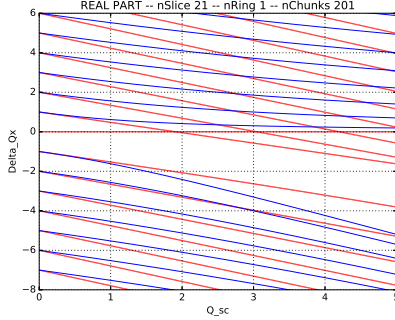


Figure 8: Positive spectrum. Simulated results (red) versus analytical results (blue).

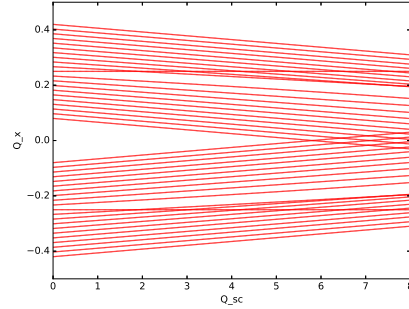


Figure 9: Positive and negative spectrum ($Q_x = 20.31$ and $Q_s = 20.017$).

Figure 10: BimBim simulated spectra with 21 slices, 1 ring and 201 chunks.

Fig.9 demonstrates an effect responsible for a limitation in the acceptable number of slices of the simulation. Since the eigenvalues of the CMM are always produced in couples with opposite signs (as mentioned in Sec.3.25), the negative part of the spectrum can meet with the positive part, and even create couplings with it. The physical reality of those "positive-negative" coupling is highly questionable since the presence of negative eigenvalues is an artifact of our analysis, which is limited to a single Poincare section of the accelerator. This effect limits strongly the maximal number of slices in situations where the fractional part of Q_s is not negligible in front of the fractional part of Q_x since the eigenvalues generated by BimBim are centered around the fractional part of $\pm Q_x$ and spaced with Q_s .

6 Convergence analysis

No formal convergence analysis of BimBim has yet been achieved in the past. The main reason being the fact that it is generally quite difficult to systematically identify the oscillation eigenmode corresponding to a certain eigenvalue of the one turn matrix generated by BimBim. This analysis becomes even more complex if one takes the radial modes into account. Nevertheless, a convergence analysis is feasible if we limit ourselves to the study of azimuthal modes for weak space charge parameter Q_{sc} . In this situation, the modes do not cross each other and it is thus possible to systematically identify the corresponding eigenmode for every eigenvalue. The situation considered here is exactly the same considered in Fig.7, with Q_{sc} chosen between 0 and 2.

Due to the absence of analytical solution of the problem, the relative error computed on Fig.11 has been computed for the 4 first modes in a bi-logarithmic scale, relatively to a solution computed with 173 slices. This *reference solution* can be considered as sufficiently close to the *converged* solution. The rate of convergence measured by a linear fit of the plot is close to -1.948 . The errors are computed for modes $-2, -1, 0, 1$ and 2 . This figure indicates that while the convergence occurs relatively early, the highest the order of the mode, the more slices it needs to reach convergence.

Unfortunately, this doesn't allow us to reliably assess the convergence of the simulation when taking radial modes into account nor when considering other effects in the one turn matrix (namely, impedance).

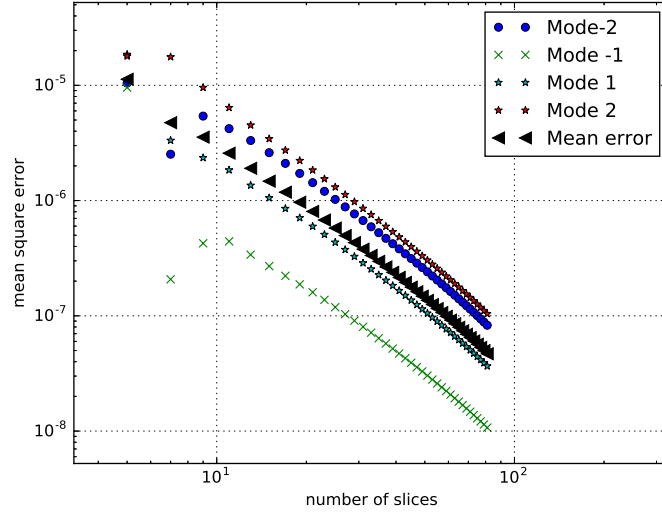


Figure 11: Mean square relative error in function of the number of slices (error computed relatively to a reference solution simulated with 173 slices).

Proton energy [Gev]	26
N_b	$0 - 6 \cdot 10^{11}$
$\epsilon_{x,y}$ [m]	$2.5 \cdot 10^{-6}$
C [m]	6900
σ_z [m]	0.23
Q_s	20.017
$Q_{x,y}$	20.13/20.18

Table 2: SPS injection parameters. [6]

7 Numerical Results: Super Proton Synchrotron

The Super Proton Synchrotron (SPS) is a circular particle accelerator currently operating at CERN. It is currently used to receive proton bunches injected from the Proton Synchrotron at 26 GeV and to accelerate them up to the energy of 450 GeV with a bunch intensity of $1.2 \cdot 10^{11}$. As part of the HL-LHC upgrade, its intensity will be increased to $2.3 \cdot 10^{11}$. Recalling the conclusions made on space charge effects in Sec. 2.5, this increase in intensity could cause stability problems during the injection phase, where the energy is relatively low (250 times lower than the current maximal energy of LHC). Researchers have thus now been investigating the possible effects on this increase of intensity on the beam stability.

As part of this investigation, A. Oeftiger simulated the spectral power density of the average position (i.e. the centroid) of an SPS bunch, considering synchro-betatron transport, impedance and space charge effects. Those results were obtained with the help of a macro-particles simulation (also called *tracking* simulation), where the space charge forces are self-consistently calculated by computing the fields generated by each macro-particle, the spectral density being then obtained by squaring the amplitude of the Fourier Transform of the macro-particles first order moment (usually denoted as *dipolar* moment). Those results have not yet been benchmarked. Being aware of the uncertain validity region of our BimBim simulation, we tried to reproduce its results in presence of weak space charge effect, i.e. in the case of a larger beam than the operational one.

Table 2 presents the operational parameters of the SPS. They were used to generate the results presented on the following figures, with the exception of an increased emittance of $8.5 \cdot 10^{-6} \text{ m}$.

Fig. 12 shows the tune shifted spectrum simulated when considering synchro-betatron motion and space charge only, considering radial and azimuthal modes. We can see that the behaviour of the space charge shifted spectrum appears to be very different from Fig. 7, indicating that the theoretical approach considering only azimuthal modes might not be in good agreement with the

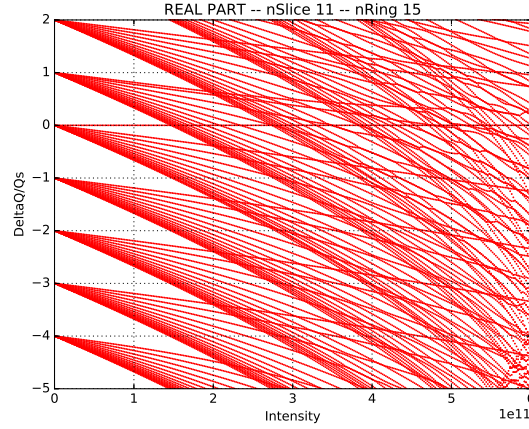


Figure 12: SPS tune spectrum generated with space charge forces and synchro-betatron transport only.

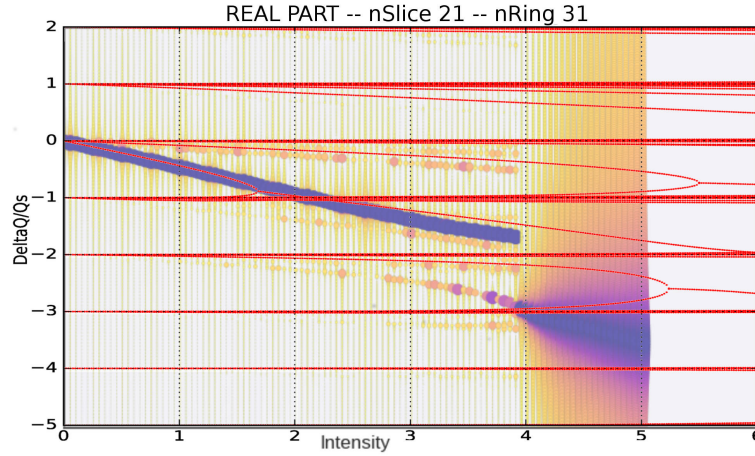


Figure 13: Comparison between spectral power density obtained by tracking simulation and BimBim spectrum (21 slices and 31 rings) for synchro-betatron transport and impedance only.

beam actual behaviour.

Fig.13 displays a comparison between the BimBim simulated spectrum and the tracking spectral power density in the case of synchro-betatron transport and impedance only. The number of slices was limited due to the effect of the negative modes mentioned in Sec.5. It shows a relative agreement between the particle in cell simulation and the BimBim simulated impedance effect. Nevertheless, the limitation in the acceptable number of slices does not allow for a full convergence. This fact underlines once more that, while being outside the scope of this project, it is crucial to solve this limitation of the CMM.

The comparison between tracking spectral power density and BimBim simulated spectrum for synchro-betatron motion, impedance and space charge effects is plotted on Fig.14. It is worth noting that the only visible modes on the tracking simulation are the one with $n_a = 0$ since it is based on the spectral analysis of the average position of the bunch. We can conclude from this comparison that both simulations present very concordant results.

The comparison of those three figures sheds light on the problem: while space charge effects dictate the general behaviour of the spectrum, the only effect of impedance in this situation appears to be the coupling between mode 0 and -1 (visible on Fig.14 around the intensity of $2 \cdot 10^{11}$). This plot can also be considered as an encouraging result concerning the validity region of our simulation. BimBim results are in agreement with the tracking simulation until an intensity of $6 \cdot 10^{11}$, this intensity corresponds in the situation of the SPS to a space charge parameter Q_{sc} close to 4.5,

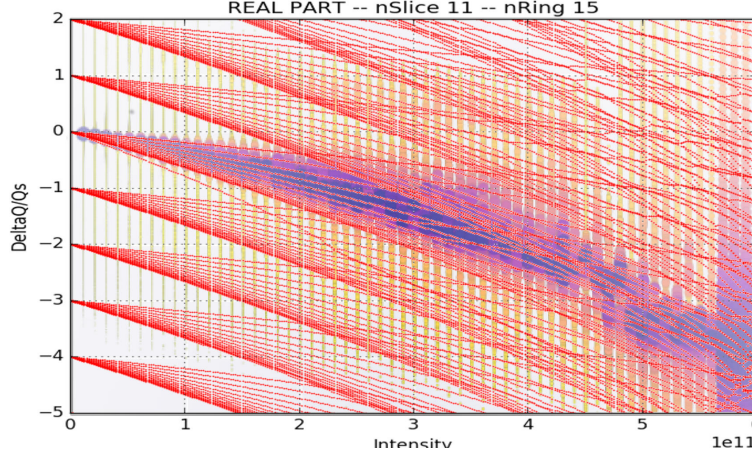


Figure 14: Comparison between spectral power density obtained by Oeftiger and BimBim spectrum for synchro-betatron transport, impedance and space charge effects.

located way above the crossing of mode 0 by mode 1. This might indicate that one of the two following statements is true: either the crossing of the positive modes with mode 0 has a physical meaning, either the validity region depends also on the number of rings we select.

Nevertheless, some disagreements appear when comparing the instabilities *growth rate* (defined as the imaginary part of the eigenvalues multiplied by the particle revolution frequency) predicted by both simulations (shown on Fig.15), recalling that we should still treat both simulation results with caution.

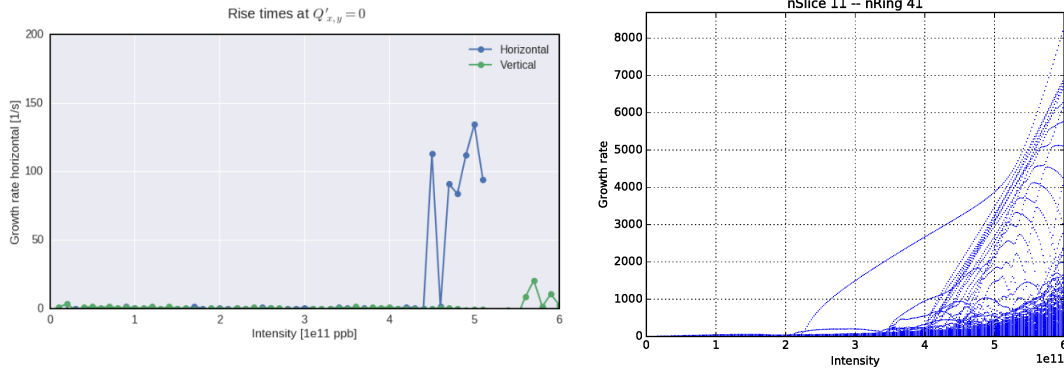


Figure 15: tracking and BimBim growth rates for synchro-betatron motion, impedance and space charge effects

8 Conclusion and prospects

In conclusion, we derived an analytical estimation of the space charge forces applied to the Circulant Matrix Model phase space decomposition considering the horizontal dimension of a single bunch. A systematic method of building the space charge matrix was then implemented as part of the BimBim semi-analytical solver. The code was then benchmarked against analytical results, but the establishment of a validity region could not be fully achieved. A basic convergence analysis was carried out and BimBim results were eventually confronted to ongoing research simulation results of the tune shifted spectrum in the case of the Super Proton Synchrotron at injection energy. Those plots were conclusive about the validity of the BimBim simulation. To conclude, the exact role of space charge forces in this situation was investigated and revealed that they were accountable for the general behaviour of the tune spectrum.

The investigation of space charge forces with the help of BimBim is obviously not concluded.

On the implementation level, one could extend the model to include the y transverse plane as well as many-bunch beam and counter circulating beams. There is probably still room for a complexity improvement of the cell retrieving algorithm. A more conclusive benchmark and an extensive convergence analysis are also highly needed. The nature of the mode crossing, as well as the impact of the negative modes needs to be understood, in order to make quantitative comparison with measurements as well as predictions for the machine upgrade.

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