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$\pi\pi$ scattering lengths and $K \rightarrow 3\pi$
decays

Candidato:

Stefano Gallorini

Relatori:

Dott. Sergio Giudici

Prof. Gino Isidori

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Chapter 1

Introduction

The phenomenology of $K \rightarrow 3\pi$ decays, pioneered by Dalitz (Ref. [1]) in 1956, has been worked over by many authors (Ref. [2–5]) during the years 1960–1970. From the assumption that dependence on kinematic variables be of polynomial form, it followed a phenomenological parametrisation for the decay rate (Ref. [3]):

$$|\mathcal{M}_{K \rightarrow 3\pi}|^2 \propto [1 + \sum_{i=1}^3 g_i \frac{(s_i - s_0)}{m_{\pi^+}^2} + h_i \frac{(s_i - s_0)^2}{m_{\pi^+}^4} + \text{higher powers}] \quad (1.1)$$
$$s_i = (P_K - P_{\pi^{(i)}})^2 \quad s_0 = \frac{1}{3} \sum_{i=1}^3 s_i = \frac{1}{3} (M_K^2 + m_{\pi^a}^2 + m_{\pi^b}^2 + m_{\pi^c}^2)$$

that is nothing but a Taylor expansion around the Dalitz plot center s_0 . The phenomenological parameters g_i, h_i are the *slope parameters* of the decay. Theoretical predictions on these parameters were made in these earlier works using isospin symmetry (Ref. [2, 3]).

The final state interaction due to strong re-scattering effects in $K \rightarrow 3\pi$ decays was considered for the first time by Sawyer and Wali (Ref. [6]) in 1960: they pointed out that the Weinberg-Taylor expansion could be invalid on the edge of the Dalitz plot because of a singularity of the decay amplitude arising from its dependence on 2π scattering amplitudes; the latter depend on the relative momentum of the pion couple (square-root behaviour). The low statistics gathered by experiments during the years 1960–70 did not permitted to reach a sufficient accuracy to detect the irregularities generated by $\pi\pi$ re-scattering and the polynomial parametrisation (1.1) was successfully employed to describe data. In the last few years a large amount of data became available for more detailed studies of $K \rightarrow 3\pi$ decays Dalitz plots. In 2003–2004 the

experiment NA48/2 at CERN SPS collected a sample of about $2.3 \cdot 10^7$ fully reconstructed $K^\pm \rightarrow \pi^\pm \pi^0 \pi^0$ decays. These data showed an anomaly in the $\pi^0 \pi^0$ invariant mass distribution ($M_{\pi^0 \pi^0}$) in the region around $M_{\pi^0 \pi^0} = 2m_{\pi^\pm}$ (Fig. 1.1). This cusp-like structure was explained theoretically by Cabibbo in terms of the re-scattering process $\pi^+ \pi^- \rightarrow \pi^0 \pi^0$ in the final state (Ref. [7]), as early suggested by Sawyer and Wali.

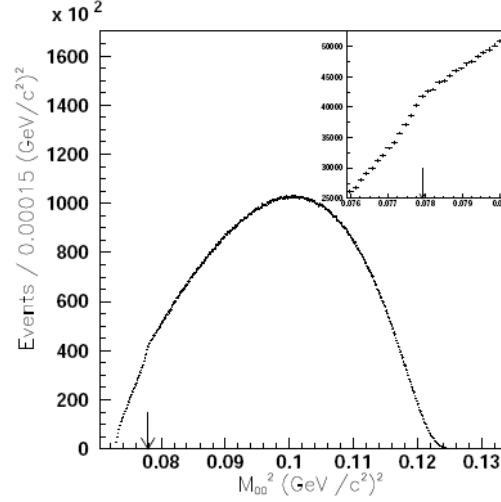


Figure 1.1: *Distribution of $M_{\pi^0 \pi^0}^2$, the square of the $\pi^0 \pi^0$ invariant mass, for the $K^+ \rightarrow \pi^+ \pi^0 \pi^0$ decays, taken from Ref. [10]. The insert is an enlargement of a narrow region centered at $M_{\pi^0 \pi^0}^2 = 4m_{\pi^\pm}^2$ (this point is indicated by the arrow). The statistical error bars are also shown in these plots*

The Cabibbo re-scattering model provided a practical method for the extraction of the $\pi\pi$ S-wave scattering lengths combination $a_{I=0}^0 - a_{I=2}^0$ and the slope parameters. Successively, the model was improved by Cabibbo and Isidori (Ref. [8]) taking into account 2-loops corrections and extending the analysis also to the $K_L^0 \rightarrow 3\pi^0$ system; in this formulation, the matrix element for $K^+ \rightarrow \pi^+ \pi^0 \pi^0$ decay includes several additional terms which depend on three S-wave scattering lengths combinations describing $\pi^+ \pi^- \rightarrow \pi^0 \pi^0$, $\pi^+ \pi^0 \rightarrow \pi^+ \pi^0$ and $\pi^0 \pi^0 \rightarrow \pi^0 \pi^0$ scattering, respectively. The fit on the $\pi^0 \pi^0$ invariant mass distribution including these re-scattering terms was implemented in 2005 by NA48/2 collaboration and a precise measurement of $a_0^0 - a_2^0$ was obtained (Ref. [9, 10]).

The $\pi\pi$ scattering lengths are a very interesting probe of QCD dynamics at low energies: their values are precisely calculable using the framework of ChPT (see Chapter 1). For this reason, the experimental determination of the

a_I^0 by using $K \rightarrow 3\pi$ is a test of QCD at low energies and, particularly, the most significant test about the basic assumptions of ChPT.

The largest error on $a_0^0 - a_2^0$ quoted by NA48/2 was the 5% theoretical uncertainty estimated naively by Cabibbo-Isidori as the result of neglecting higher-order terms and radiative corrections in their re-scattering model. The first aim of this thesis is to include 3-loops diagrams in the model, in order to give a more precise estimation of the theoretical error. A posteriori checks about the size of this error can be obtained by studying the experimental central values of $a_0 - a_2$ obtained with and without 3-loops corrections. The 2005 result on $a_0 - a_2$ stimulated the re-analysis of the old $K_L^0 \rightarrow 3\pi^0$ data taken eight years ago during the 2000 by the NA48 experiment at CERN SPS in the perspective of the cusp effect. These data were used in the 2001 by the NA48 collaboration for the measurement of the quadratic slope parameter h in the $K_L^0 \rightarrow 3\pi^0$ decay Dalitz plot (Ref. [12]) according to Eq. (1.1) and Particle Data Group parametrisation (Ref. [13]), i.e. without re-scattering. The cusp structure in K_L^0 system is less visible than in $K^+ \rightarrow \pi^+\pi^0\pi^0$ decay but it is not negligible. Moreover, rescattering effects change the determination of the slope parameter h . In the second part of the thesis, we implement the cusp model in the analysis of $K_L^0 \rightarrow 3\pi^0$ data, providing an independent determination of $a_0 - a_2$ and extracting the value of h corrected for rescattering effects.

The thesis is divided in two parts (**part I**, **part II**) and organized as follows. In **part I**, **Chapter 2**, we introduce Chiral Perturbation Theory and the basic theoretical tools to describe $\pi\pi$ scattering. As an application, we discuss in detail the first-order calculation of a_0 and a_2 . More accurate results, obtained by taking into account higher-order corrections, are also given at the end of the chapter. In **Chapter 3**, we review the Cabibbo-Isidori model for $K^+ \rightarrow \pi^+\pi^0\pi^0$ and $K_L^0 \rightarrow 3\pi^0$ systems; in order to clarify the general strategy adopted, we give some examples of 1-loop and 2-loops calculations. The 3-loops corrections to the matrix element are discussed in **Chapter 4**, where detailed calculations are given. Moreover, we present a preliminary estimation of the theoretical error based on 3-loops corrections. The results and conclusions of part I are summarized in **Chapter 5**.

In **part II**, **Chapter 6**, we list the recent measurements of $a_0 - a_2$ and slope h and we discuss in brief the underlying physical motivations and the methods used to extract these quantities.

Chapter 7 is devoted to describe the NA48 experiment and the detectors needed for collecting the $K_L^0 \rightarrow 3\pi^0$ events; furthermore, we give a general

description of the neutral trigger and trigger efficiency. The data reconstruction procedure is presented in **Chapter 8**. Data to Monte Carlo comparison and the detector acceptance are also studied. In the first part of **Chapter 9**, the general strategy of the fitting analysis is outlined; in the second part of the chapter, we give the results obtained for the parameters by using the leading order and the next-to-leading order approximation. In **Chapter 9** we quote the final results and we give the conclusions of part II. Two appendixes complete the thesis: the isospin relation and $\Delta I = 1/2$ rule in $K \rightarrow 3\pi$ (Appendix A) and the unitarity and analyticity of S matrix (Appendix B), where the Cutkosky rules are briefly presented.

Part I

$\pi\pi$ scattering lengths and final
state interaction in $K \rightarrow 3\pi$ decays

Chapter 2

Chiral Perturbation Theory and $\pi\pi$ scattering lengths

In this chapter we introduce Chiral Perturbation Theory (ChPT), the low-energy effective theory of strong interactions. In this framework we derive the values of the S -wave scattering Lengths $a_{I=0}^0$, $a_{I=2}^0$ at the leading order, reporting also the more precise results obtained by using higher-orders calculations.

2.1 The basis of Chiral Perturbation Theory

The theory of the strong interactions is Quantum Chromodynamics (QCD), a theory formulated in terms of quarks and gluon fields built on the principle of gauge invariance with the color gauge group $SU(3)_c$. The large strong coupling constant at low energies (a consequence of the phenomena of asymptotic freedom) prevents a use of perturbation theory, so there is no direct link between QCD and the fundamental degrees of freedom and the relevant hadronic degrees of freedom as observed in the spectrum of mesons and baryons. An alternative approach for studying the strong interaction at low energies is to construct an effective theory that deal from the beginning with the mesons and baryons (Ref. [14]).

The basic idea of an effective theory is the following: as long as one is only interested in a particular range of distances or energies, scales much bigger or much smaller than this range should not influence the description of the system in question too strongly. In particular, the dynamics at low energies is supposed to be independent on the details of the dynamics at high energies. As a result, low energy physics can be described using a effective lagrangian that contains only the relevant degrees of freedom, ignoring the heavy ones and that shares

the symmetries of the underlying theory. In order to construct the effective theory for the strong interactions, we have to investigate the symmetries of the QCD lagrangian. For this purpose, we decompose the quark field into its chiral components:

$$q = \frac{1}{2}(1 - \gamma_5)q + \frac{1}{2}(1 + \gamma_5)q \equiv q_L + q_R$$

The QCD lagrangian is then:

$$\begin{aligned}\mathcal{L}_{\text{QCD}} &= \mathcal{L}_{\text{QCD}}^0 + \mathcal{L}_{\text{QCD}}^m + \dots, \\ \mathcal{L}_{\text{QCD}}^0 &= \frac{1}{2}\text{Tr} G_{\mu\nu}^a G^{a,\mu\nu} + i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R \\ \mathcal{L}_{\text{QCD}}^m &= \bar{q}_L \mathcal{M} q_R + \bar{q}_R \mathcal{M}^\dagger q_L\end{aligned}\tag{2.1}$$

where D_μ is the covariant derivative, $G_{\mu\nu}^a$ is the gluon field strength tensor, q collects the light quark flavours $q^T = (u, d, s)$ and $\mathcal{M} = \text{diag}(m_u, m_d, m_s)$ is the quark mass matrix; the ellipses denotes the heavier quark flavours. In the limit of vanishing quark masses, $\mathcal{L}_{\text{QCD}}$ displays the $U(3)_L \times U(3)_R$ *chiral symmetry*:

$$(q_R, q_L) \rightarrow (Lq_L, Rq_R), \quad L, R \in U(3)_{L,R}\tag{2.2}$$

The masses of the three light quarks are small compared to the typical hadronic scale,

$$m_{u,d,s} \ll 1\text{GeV} \approx \Lambda_{\text{hadr.}}$$

The first assumption of Chiral Perturbation Theory (Ref. [15–17]) is that we can perform a perturbative expansion around the chiral limit ($m_u = m_d = m_s = 0$). The chiral group can be rewritten according to:

$$U(3)_L \times U(3)_R = SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A$$

where we have introduced the vector $V = L + R$ and the axial vector $A = L - R$ transformations. The Noether current associated with $U(1)_V$ is conserved in the standard model while the $U(1)_A$ current is broken by quantum effects, the $U(1)_A$ anomaly (Ref. [18]). The $SU(3)_L \times SU(3)_R$ currents are certainly broken by the mass term, but this expected to be a small effect. There are two ways to realize a global chiral symmetry: in the *Wigner mode*, where the symmetry is exact and realized linearly on fields or in the *Goldstone mode*, where the symmetry is spontaneously broken (SSB) and realized non-linearly. The

Wigner mode realization of chiral symmetry would lead to a parity doubling in the hadron spectrum that is not observed: we find (approximate) $SU(3)_V$ multiplets and no parity doubling is observed. Several theoretical and numerical studies suggest that chiral symmetry is spontaneously broken and, in accordance with the observations of hadronic multiplets, the breaking pattern is (Ref. [19]):

$$SU(3)_L \times SU(3)_R \rightarrow SU(3)_V$$

The axial charges commute with the Hamiltonian, but don't leave the ground state invariant. As a consequence, 8 pseudoscalars Goldstone bosons appear. In first approximation these can be identified with:

$$\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$$

The Goldstone particles have the important property that their interaction become arbitrarily weak at low momenta: in a SSB theory, we may always choose a group transformation (near to the identity) $\gamma(x) \in SU(3)_L \times SU(3)_R$ (space-time dependent) acting on a field $\tilde{\psi}(x)$ from which the massless Goldstone modes has been eliminated (in general, $\tilde{\psi}$ include all the heavy fields of the theory) (Ref. [20]):

$$\psi_n(x) = \sum_m \gamma(x)_{nm} \cdot \tilde{\psi}_m(x) \quad (2.3)$$

The Goldstone bosons whose fields are eliminated in this way will reappear as fields needed to parametrize $\gamma(x)$ ¹. Since the lagrangian is invariant only under spacetime-independent transformations, it follows that the transformation (2.3) depends on $\gamma(x)$ with at least one derivative in each γ -dependent term. As a result, the coupling of the Goldstone bosons vanish in the limit of vanishing momenta. In order to build a low-energy effective theory of the strong interactions at low energies, we must construct the most general lagrangian in terms of the Goldstone fields, invariant under the chiral symmetry (in this chapter we deal only with the eight Goldstone bosons and we don't include heavier states such as baryons, in the effective lagrangian). The general formalism for effective lagrangians for spontaneously broken symmetries was worked out by Callan, Coleman, Wess and Zumino (Ref. [21]). Using the polar decomposition for $\gamma(x)$, the Goldstone fields $\pi^a(x)$ are parametrized as²:

¹In general, it can be shown that γ belongs to the right coset G/H , where H is the unbroken subgroup ($\dim(G/H) = \dim(G) - \dim(H)$).

²The choice of the parametrization of γ and the representation of the group is arbitrary

$$U(\phi(x)) \equiv \gamma^2(x) \equiv \exp \left\{ i \frac{\sqrt{2}\phi(x)}{F} \right\} \quad (2.4)$$

where:

$$\phi = \phi^a T^a = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix} \quad (2.5)$$

T^a , $a = 1..8$ are the Gell-Mann matrices, chosen as generators of the broken group and F is a dimensional constant (to be determined). Under the chiral group, $\phi(x)$ transforms as:

$$g_0 \cdot \gamma(\phi(x)) = \gamma(\phi'(x)) \cdot h(\phi(x), g_0) \quad (2.6)$$

where $g_0 \in SU(3)_L \times SU(3)_R$ and $h \in SU(3)_V$. From the invariance of the chiral group under parity ($SU(3)_R \leftrightarrow SU(3)_L$) it follows that

$$U(\phi) \rightarrow g_L U(\phi) g_R^\dagger \quad (2.7)$$

where $g_{L,R} \in SU(3)_{R,L}$.

The most general lagrangian invariant under (2.7) must be product of terms of the form ³ $\text{Tr}(\partial_\mu U \partial^\mu U^\dagger \dots \partial_\nu U \partial^\nu U^\dagger)$. The only invariant term with two derivative is:

$$\mathcal{L}_2 = \frac{F^2}{4} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) \quad (2.8)$$

Expanding U in power series in ϕ , we have:

$$\mathcal{L}_2 = \frac{1}{2} \text{Tr}(\partial_\mu \phi \partial^\mu \phi) + \frac{1}{6F^2} \text{Tr}[\phi, \partial_\mu \phi]^2 + \dots \quad (2.9)$$

The constant F can be determined by the matrix element of the axial current and pion states: $\langle 0 | j_A^{\mu,a} | \pi^b \rangle = i F_\pi p^\mu \delta^{ab}$. The result is: $F = F_\pi \sim 93 \text{ MeV}$, the pion decay constant.

³Terms such as $\text{Tr}(U U^\dagger \dots U U^\dagger)$ are constant ($U U^\dagger = 1$).

The chiral symmetry is explicitly broken by the quarks mass matrix so that a breaking term must be added to Eq. (2.8):

$$\mathcal{L}_m = B \frac{F_\pi^2}{2} \text{Tr} (U^\dagger \mathcal{M} + \mathcal{M}^\dagger U) \quad (2.10)$$

where, according to the Gell-Mann-Oakes-Renner relation (Ref. [22]), we can identify the free parameter B with $B = -\langle 0 | \bar{\psi}\psi | 0 \rangle / F_\pi^2$. This term is linear in the quark masses and does not contain derivatives. Expanding U to the second order in π , we can find a non vanishing mass for the pion:

$$m_\pi^2 = B (m_u + m_d)$$

Finally, the full chiral lagrangian at $O(p^2)$ and at the first order in the quark masses is:

$$\mathcal{L}_2 = \frac{F_\pi^2}{4} \text{Tr} (\partial_\mu U \partial^\mu U^\dagger) + B \frac{F_\pi^2}{2} \text{Tr} (U^\dagger \mathcal{M} + \mathcal{M}^\dagger U) \quad (2.11)$$

2.2 $\pi\pi$ scattering

An application of the ChPT at p^2 order is the low energy scattering $\pi^a \pi^b \rightarrow \pi^c \pi^d$. Expanding Eq. (2.11) to fourth order in ϕ , we have:

$$\mathcal{L}_2 = \frac{1}{6 F_\pi^2} \text{Tr} [\phi, \partial_\mu \phi]^2 + B \frac{2}{3} \text{Tr} (M \phi^4) + \dots \quad (2.12)$$

The invariant amplitude $T_{a,b;c,d} \equiv \langle p^c, p^d | T | p^a, p^b \rangle$ is determined by a single function $A(s, t, u)$:

$$T_{a,b;c,d} = \delta_{ab} \delta_{cd} A(s, t, u) + \delta_{ac} \delta_{bd} A(t, s, u) + \delta_{ad} \delta_{bc} A(u, t, s) \quad (2.13)$$

where a, b, c, d label the isospin components and s, t, u are the Mandelstam variables:

$$\begin{aligned} s &= 4(k^2 + m_\pi^2) \\ t &= -2k^2(1 - \cos \theta) \\ u &= 4m_\pi^2 - s - t \end{aligned}$$

k is the relative momentum and θ the scattering angle in the C.M. frame.

Using Eq. (2.12) we have:

$$A(s, t, u) = \frac{s - m_{\pi^+}^2}{F_\pi^2} \quad (2.14)$$

The T^I -matrix with definite isospin can be expanded in partial waves amplitudes (Ref. [23]):

$$T^I(s, t, u) = 32\pi \sum_{\ell} (2\ell + 1) P_{\ell}(\cos \theta) T_{\ell}^I(k) \quad (2.15)$$

$$T_{\ell}^I(k) = \frac{e^{2i\delta_{\ell}^I(k)} - 1}{2ik} \quad (2.16)$$

$$e^{2i\delta_{\ell}^I(E)} \delta_{\ell\ell'} \delta_{II'} \delta(E' - E) = \langle E', I', \ell' | S | E, I, \ell \rangle \quad (2.17)$$

where $\delta_{\ell}^I(k)$ is the phase-shift.

For low-energy scattering ($k \rightarrow 0$) (Ref. [23]):

$$T_{\ell}^I \sim k^{2\ell} a_{\ell}^I + \dots \quad (2.18)$$

The quantities a_{ℓ}^I are referred to as the $\pi\pi$ scattering lengths.

In order to extract the scattering lengths, we need T^I as combinations of $A(s, t, u)$:

$$T^{I=0} = 3A(s, t, u) + A(t, u, s) + A(u, s, t)$$

$$T^{I=1} = A(t, u, s) - A(u, s, t)$$

$$T^{I=2} = A(t, u, s) + A(u, s, t)$$

Using Eq. (2.14) and (2.19) at threshold $s = 4m_{\pi}^2$ ($l = 0$), we find ⁴:

$$a_0^0 = \frac{7m_{\pi}}{32\pi F_{\pi}^2} = 0.16 m_{\pi^{\pm}}^{-1} \quad (2.19)$$

$$a_0^2 = -\frac{m_{\pi}}{16\pi F_{\pi}^2} = -0.045 m_{\pi^{\pm}}^{-1} \quad (2.20)$$

This is the Weinberg result (Ref. [24]). Various improvements of this result have been obtained by Gasser and Leutwyler and other authors (Ref. [25] - [28]). At present, the scattering lengths are computed at $\mathcal{O}(p^6)$ accuracy and the updated predictions are:

⁴In the following, we will use $m_{\pi^{\pm}}$ as unit of energy for the scattering lengths, so that we write e.g. a_0^0 instead of $a_0^0 m_{\pi^{\pm}}$.

$$\begin{aligned}
a_0 &\equiv a_0^0 = 0.220 \pm 0.005 \\
a_2 &\equiv a_2^0 = -0.0444 \pm 0.001
\end{aligned} \tag{2.21}$$

It was pointed out by Morgan, Shaw (Ref. [29]) and Colangelo, Gasser and Leutwyler (Ref. [11], [28]) that the possible values of a_2 and a_0 are restricted to a band in the (a_0, a_2) plane, the so-called universal band (UB).

This band is defined as the area which is allowed by $\pi\pi$ scattering data above 0.8 GeV, dispersion relations and chiral symmetry constraints; the central curve of this band is given by:

$$a_2 = -0.0849 + 0.232 a_0 - 0.0865 a_0^2 [\pm 0.0088]$$

a_2 and a_0 are then related as:

$$\Delta a_2 = 0.236 \Delta a_0 - 0.61 (\Delta a_0)^2 - 9.9 (\Delta a_0)^3 \tag{2.22}$$

where $\Delta a_0 = a_0 - 0.220$ and $\Delta a_2 = a_2 + 0.0444$.

Chapter 3

$\pi\pi$ scattering and $K^+ \rightarrow \pi^+\pi^0\pi^0$ decay

In this chapter we illustrate the $\pi\pi$ rescattering model in $K \rightarrow 3\pi$ decays proposed by Cabibbo and Isidori (Ref. [7], [8]) to explain the anomalous behaviour in $\pi^0\pi^0$ invariant mass spectrum from $K^+ \rightarrow \pi^+\pi^0\pi^0$ decay, observed in 2004 by NA48/2 collaboration (Ref. [9], [10]). This model provides a perturbative expansion about the $I=0,2$ scattering lengths a_0 and a_2 that are assumed to be small on the grounds of ChPT predictions and available experimental results (Chapter 6). This expansion is equivalent to an expansion in v_{ij} , the relative velocity of the pions $\pi^i\pi^j$ arising as intermediate state in the $K \rightarrow 3\pi$ decay (Fig. 3.1). The properties of unitarity (optical theorem) and analyticity of the $S_{\pi\pi \rightarrow \pi\pi}$ matrix allow, through the Cutkosky rules (Ref. [30] and Appendix B), to perform the expansion in a systematic way. In the following sections we list the 1-loop and 2-loops diagrams and the results obtained by Cabibbo-Isidori for $K^+ \rightarrow \pi^+\pi^0\pi^0$ and $K_L^0 \rightarrow 3\pi^0$ decays. 3-loops calculations for the K^+ system will be discussed in the next chapter.

3.1 Amplitude decomposition

Following Ref. [8] the matrix element for $K^+ \rightarrow \pi^+\pi^0\pi^0$ decay can be decomposed as:

$$\mathcal{M}_{+00} \equiv \mathcal{A}_{+00} + \mathcal{B}_{+00} v_{\pm}(s_3) \quad (3.1)$$

and

$$\mathcal{M}_{+00} \equiv \begin{cases} \text{Re}\mathcal{A}_{+00} + i \text{Im}\mathcal{A}_{+00} + (\text{Re}\mathcal{B}_{+00} + i \text{Im}\mathcal{B}_{+00}) v_{\pm}(s_3) & s_3 > 4m_{\pi^+}^2 \\ \text{Re}\mathcal{A}_{+00} + i \text{Im}\mathcal{A}_{+00} + (-\text{Im}\mathcal{B}_{+00} + i \text{Re}\mathcal{B}_{+00}) v_{\pm}(s_3) & s_3 < 4m_{\pi^+}^2 \end{cases} \quad (3.2)$$

where

$$s_i = (P_K - P_{\pi_i})^2, \quad i = 1, 2, 3$$

are the Mandelstam variables and

$$v_{\pm}(s_3) = \sqrt{\frac{|s_3 - 4m_{\pi^+}^2|}{s_3}}$$

is the absolute value of the $\pi^+\pi^-$ relative velocity (in C.M. frame); the index ‘3’ labels the “odd” π^+ pion so that $s_3 = M_{\pi^0\pi^0}^2$ is the $\pi^0\pi^0$ invariant mass.

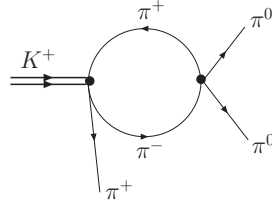


Figure 3.1: The re-scattering diagram with the $\pi^+\pi^- \rightarrow \pi^0\pi^0$ charge-exchange reaction; in this case, the $K^+ \rightarrow \pi^-\pi^+\pi^+$ decay appears as intermediate state

The amplitudes $\mathcal{A}_{+00}$ and $\mathcal{B}_{+00}$ are defined to contain only terms with even powers of $v_{\pm}$ ($(v_{\pm})^{2k}$, $k = 0, 2, 4, \dots$) i.e. they are analytic functions at the threshold point $s_3 = 4m_{\pi^+}^2$. This parametrization gives emphasis to the discontinuity arising at $s_3 = 4m_{\pi^+}^2$ where the $\pi^+\pi^-$ velocity flips from real (above cusp) to imaginary (below cusp): in the final state interaction $\pi^0\pi^0 \rightarrow \pi^0\pi^0$, the $\pi^+\pi^-$ intermediate state is forbidden by energy conservation as long as $s_3 < 4m_{\pi^+}^2$; when this state become accessible ($s_3 > 4m_{\pi^+}^2$ and $v_{\pm}$ real), $S_{\pi^0\pi^0 \rightarrow \pi^0\pi^0}$ acquires a contribute to imaginary part (optical theorem) and a discontinuity arises. This non-analytic behaviour (occurring when a new channel opens) is observed as a cusp-like structure in the $\pi^0\pi^0$ invariant mass spectrum. However, irregular terms containing square-root singularities also occur in $\mathcal{A}_{+00}$ and $\mathcal{B}_{+00}$, particularly at the $\pi^0\pi^0$ and $\pi^+\pi^0$ thresholds.

In the following calculations the $K^+ \rightarrow \pi^-\pi^+\pi^+$ decay can appear as intermediate state (Fig. 3.1) so that also the amplitude $\mathcal{M}_{++-}$ must be parametrized in a similar way:

$$\mathcal{M}_{++-} = \mathcal{C}_{++-} + \mathcal{D}_{++-}^{(1)} v_{00}(s_1) + \mathcal{D}_{++-}^{(2)} v_{00}(s_2)$$

The matrix $\mathcal{M}_{+00}$ squared is then:

$$\begin{aligned} |\mathcal{M}_{+00}|^2 &= (\text{Re}\mathcal{A}_{+00})^2 + (\text{Im}\mathcal{A}_{+00})^2 + [(\text{Re}\mathcal{B}_{+00})^2 + (\text{Im}\mathcal{B}_{+00})^2] v_{\pm}^2 \\ &+ 2 v_{\pm} \Delta_{\text{cusp.}} \end{aligned} \quad (3.3)$$

where

$$\Delta_{\text{cusp.}} \equiv \begin{cases} \text{Re}\mathcal{A}_{+00} \text{Re}\mathcal{B}_{+00} + \text{Im}\mathcal{A}_{+00} \text{Im}\mathcal{B}_{+00} & s_3 > 4m_{\pi^+}^2 \\ \text{Im}\mathcal{A}_{+00} \text{Re}\mathcal{B}_{+00} - \text{Re}\mathcal{A}_{+00} \text{Im}\mathcal{B}_{+00} & s_3 < 4m_{\pi^+}^2 \end{cases} \quad (3.4)$$

As pointed out by Cabibbo-Isidori, the amplitudes $\text{Re}\mathcal{A}_{+00} \equiv R^0$ and $\text{Re}\mathcal{C}_{++-} \equiv R^+$ can be well approximated as polynomials on the whole physical range, having in mind that near the $\pi^0\pi^0$ and $\pi^+\pi^0$ thresholds, the polynomial form is not appropriate to describe the square-root singularities; according to the old pre-rescattering $K \rightarrow 3\pi$ parametrization (Ref. [3], [5], [13]), R^0 and R^+ can be expanded in a Taylor series about their respective Dalitz plot centers:

$$R^0(s_3) = A^0 [1 + g^0 (s_3 - s_0^0)/2 m_{\pi^+}^2 + \tilde{h}^0 (s_3 - s_0^0)^2/2 m_{\pi^+}^4] \quad (3.5)$$

$$R^+(s_3) = A^+ [1 + g^+ (s_3 - s_0^+)/2 m_{\pi^+}^2 + \tilde{h}^+ (s_3 - s_0^+)^2/2 m_{\pi^+}^4] \quad (3.6)$$

where $g^0, \tilde{h}^0$ and $g^+, \tilde{h}^+$ are called the *slope parameters* of the $K^+ \rightarrow \pi^+\pi^0\pi^0$ and $K^+ \rightarrow \pi^+\pi^+\pi^-$ decays, respectively. These slopes are expected to be slightly different from the quoted PDG values so that they should be measured experimentally according to the new rescattering parametrisation. A^0, A^+ are the isospin amplitudes for the two decays; isospin symmetry and $\Delta I = 1/2$ rule predicts (see Appendix A):

$$\frac{A^+}{A^0} = 2$$

The centers of the Dalitz plot $s_0^{(0,+)}$ are defined as:

$$s_0^0 = \frac{1}{3} \sum_{i=1}^3 s_i = \frac{1}{3} (M_{K^+}^2 + 2m_{\pi^0}^2 + m_{\pi^+}^2)$$

$$s_0^+ = \frac{1}{3} \sum_{i=1}^3 s_i = \frac{1}{3} (M_{K^+}^2 + 3m_{\pi^+}^2)$$

It should be noted that terms such as $(s_2 - s_0^{(0,+)})$ or $(s_1 - s_0^{(0,+)})$ cannot exist in the expansions because of the identity of $\pi^0\pi^0$ or $\pi^+\pi^+$; in fact, terms linear in s_1, s_2 can appear only with the same slope coefficient j :

$$j^{(0,+)} [(s_1 - s_0^{(0,+)}) + (s_2 - s_0^{(0,+)})] = -j^{(0,+)} (s_3 - s_0^{(0,+)})$$

that can be absorbed in the linear slopes $g^{(0,+)}$.

The PDG tables (Ref. [13]) also include terms proportional to $(s_{(1,2)} - s_0^{(0,+)})^2$, but their coefficients are small and compatible with zero.

The $\pi\pi \rightarrow \pi\pi$ scattering amplitude

$$\langle \pi^d \pi^c | T | \pi^b \pi^a \rangle \equiv i \mathcal{M}_{ij}(s, t, u)$$

$(i, j) = (00, 00), (+0, +0), (+-, 00) \equiv x, (+-, +-), (++, ++)$, is a complex quantity, but in the low energy regime, near the threshold $s = 4m_{\pi^+}^2$ ($u = t \sim 0$), just its real part may be considered (optical theorem) and any dependence on the t, u variables can be neglected. The scattering lengths are then defined as the coefficients in the power expansion of the T-matrix $\langle I, I'_3 | T | I, I_3 \rangle$ in the relative momentum k , so that (Ref. [8]):

$$\text{Re } \mathcal{M}_{ij}(s) \equiv \frac{8m_{\pi^+}}{\pi} a_{ij}(s) \quad (3.7)$$

and $a_{ij}(s)$ comes from the isospin decomposition of the amplitudes:

$$a_{ij}(s) = \begin{cases} a_{00}(s) &= \frac{a_0(s) + 2a_2(s)}{3} \\ a_{+0}(s) &= \frac{a_2(s)}{2} \\ a_x(s) &= \frac{a_0(s) - a_2(s)}{3} \\ a_{+-}(s) &= \frac{2a_0(s) + a_2(s)}{6} \\ a_{++}(s) &= a_2(s) \end{cases} \quad (3.8)$$

$$a_{I=0,2}^0(s) = \begin{cases} a_0(s) &= a_0 \cdot [1 + r_0 \frac{(s - s_{\text{th.}})}{4m_{\pi^+}^2}] \\ a_2(s) &= a_2 \cdot [1 + r_2 \frac{(s - s_{\text{th.}})}{4m_{\pi^+}^2}] \end{cases} \quad (3.9)$$

where $r_0 = 1.25 \pm 0.04, r_2 = 1.81 \pm 0.05$ are the effective ranges calculated in Ref. [31] and $s_{\text{th.}} = 4m_{\pi^+}^2$. The scattering lengths a_0 and a_2 are defined so that $a_{ij}(s_{\text{th.}})$ is the value it would be in the limit of exact isospin symmetry.

3.2 General strategy of the a_i expansion

In this section we illustrate the Cabibbo-Isidori method for calculating the $K \rightarrow 3\pi$ amplitudes when the $\pi\pi$ final state interaction is turned on. For $K^+ \rightarrow \pi^+\pi^0\pi^0$ decay, we perform explicit calculations of 1-loop and 2-loops diagrams by using the Cutkosky rules (Appendix B). Moreover, we give a systematic way to expand the matrix element in powers of scattering lengths a_i (or equivalently, in powers of velocities v_i).

3.2.1 1-loop diagrams

The 1-loop diagrams have only one rescattering vertex, so they are of a_i order (fig.3.2); these diagrams can be calculated exactly giving a contribute of a_i order to both real and imaginary part of the amplitude.

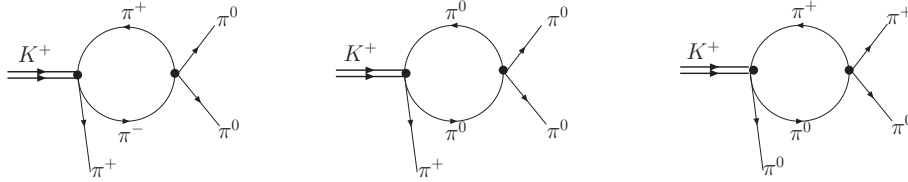


Figure 3.2: *The re-scattering diagrams generating the imaginary part at the a_i order*

The $\pi\pi \rightarrow \pi\pi$ rescattering vertices are described by Eq. (3.7) (we work near the threshold), while the $K \rightarrow 3\pi$ tree amplitudes are parametrized by Eq. (3.5)- (3.6). Following Cabibbo-Isidori, we compute the discontinuities of the 1-loop diagrams across their respective branch cuts by using the Cutkosky rules. As we need the discontinuity associated with the $s = (m_{\pi^a} + m_{\pi^b})^2$ thresholds ($a, b = \pm, 0$), we have to cut the two internal lines (propagators) and put them on shell. As an example, we calculated in detail the diagram with the $\pi^+\pi^- \rightarrow \pi^0\pi^0$ charge-exchange reaction (Fig. 3.3). In the $\pi^0\pi^0$ C.M. frame, we obtain:

$$\text{Disc}(s_3) = \int \frac{d^3p_1}{2E_1} \frac{d^3p_2}{2E_2} \delta^4(p_1 + p_2 - q_1 + q_2) i R^+((p_1 + q_3))^2 \cdot i \frac{8}{\pi} a_x((q_1 + q_2))^2$$

where we have used the property that R^+ depends only on $(P_K - P_{\pi^-})^2$.

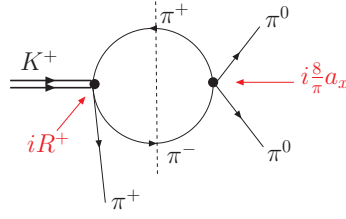


Figure 3.3: The 1-loop rescattering diagram with the $\pi^+\pi^- \rightarrow \pi^0\pi^0$ charge-exchange reaction. The vertices prescriptions and the 2π cut are also shown in the figure

At this point, we assume that the slope parameters, that one would find experimentally with this new parametrisation, should not differ appreciably from the old PDG slopes. From PDG we obtain $\tilde{h}^+ \sim 0.0004$ so that we can neglect the term $\propto (s_3 - s_0^+)^2$ in R^+ and with a good accuracy we have:

$$\text{Disc}(s_3) = -4 a_x(s_3) R^+(\langle s_3 \rangle) v_{\pm}(s_3)$$

where:

$$\langle s_i \rangle = \frac{3s_0^+ - s_i}{2}$$

The part of the amplitude that can be calculated by Cutkosky rules is:

$$i \mathcal{M}_{+00}^{\text{Cut.}} = \frac{1}{2} \text{Disc}(s_3)$$

Then, the diagram of Fig. 3.3 has value:

$$\mathcal{M}_{+00}^{\text{fig.3.3}} = \begin{cases} K(s_3) + i \cdot 2 a_x(s_3) R^+(\langle s_3 \rangle) v_{\pm}(s_3) & s_3 > 4m_{\pi^+}^2 \\ K(s_3) - 2 a_x(s_3) R^+(\langle s_3 \rangle) v_{\pm}(s_3) & s_3 < 4m_{\pi^+}^2 \end{cases} \quad (3.10)$$

$K(s_3) \propto a_i$ is the real analytic part of $\mathcal{M}_{+00}^{\text{fig.3.3}}$ and it can be expanded in powers of s_3 . This term is included in R^0 (through the slope parameters), so that we can put $K(s_3) = 0$.

Finally, we have the result:

$$\begin{aligned} \mathcal{M}_{+00}^{\text{fig.3.3}} &= \text{Im} \mathcal{B}_{+00}^{(a_i)} \\ &= 2 a_x(s_3) R^+(\langle s_3 \rangle) \end{aligned} \quad (3.11)$$

The charge-exchange diagram contribute only to imaginary part of $\mathcal{B}_{+00}$. The remaining 1-loop diagrams can be computed in the same way. We note that

the pions running inside the loop are $(\pi^0\pi^0)$ and $(\pi^+\pi^0)$ (Fig. 3.2) then these diagrams contribute only to $\text{Im}\mathcal{A}_{+00}$:

$$(\text{Im}\mathcal{A}_{+00}^{(a_i)})_{\pi^0\pi^0} = a_{00}(s_3) R^0(s_3) v_{00}(s_3)$$

$$(\text{Im}\mathcal{A}_{+00}^{(a_i)})_{\pi^+\pi^0} = \text{Im}\mathcal{A}_{+00}^{(1)} + \text{Im}\mathcal{A}_{+00}^{(2)} \equiv \text{Im}\mathcal{A}_{+00}^{(1)} + (s_1 \leftrightarrow s_2)$$

where:

$$\text{Im}\mathcal{A}_{+00}^{(1)} = 2a_{+0}(s_1) v_{+0}(s_1) R^0(\langle s_1 \rangle - \Delta_1)$$

$$\Delta_i = \frac{(m_{\pi^+}^2 - m_{\pi^0}^2)(m_K^2 - m_{\pi^0}^2)}{2s_i}$$

3.2.2 2-loops diagrams

There are two topologically distinct 2-loops diagrams: one-channel and two channel re-scattering graphs. These diagrams give a a_i^2 contribution to $\mathcal{M}_{+00}$ (two re-scattering vertices). In Fig. 3.4 all the diagrams giving $\mathcal{B}_{+00}^{(a_i^2)}$ terms ($\propto v_{\pm}$) are shown:

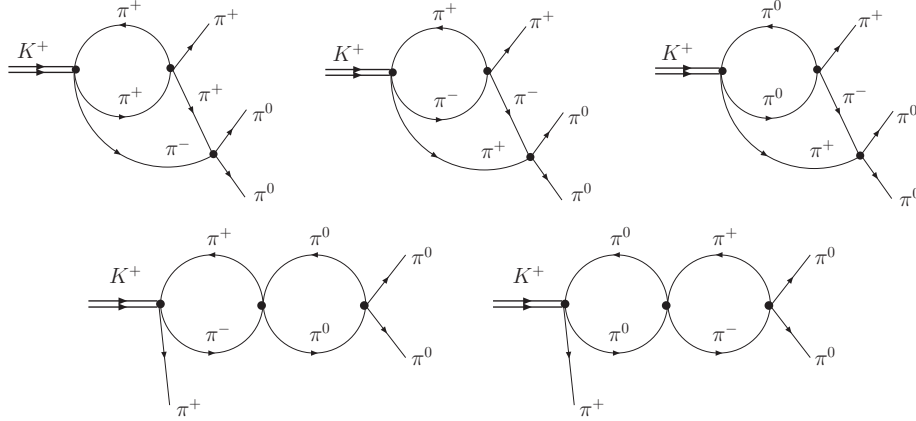


Figure 3.4: *The 2-loops diagrams that contribute to $\text{Re}\mathcal{B}_{+00}$ at the a_i^2 order: only these diagrams have the cusp factor $v_{\pm}$*

As in the 1-loop case, we illustrate the method with a detailed example calculating the diagram in Fig. 3.5. The discontinuity associated with the $s_3 = 4m_{\pi^+}^2$ threshold is given by the cut shown in Fig. 3.5.

We have:

$$i \mathcal{M}_{+00}^{Cut.} = \frac{1}{2} \int \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2} \delta^4(p_1 + p_1 - q_1 + q_2) i (\mathcal{M}_{++-}^{1 \text{ loop}})_{\pi^+\pi^-} ((p_1 + q_3)^2) \cdot i \frac{8}{\pi} a_x ((q_1 + q_2)^2)$$

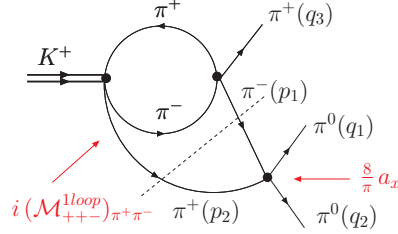


Figure 3.5: The 2-loops diagrams that contribute to $\text{Re } \mathcal{B}_{+00}$ at the d_i^2 order: only these diagrams have the cusp factor $v_{\pm}$

$(\mathcal{M}_{++-}^{1 \text{ loop}})_{\pi^+\pi^-}$ follows from 1-loop analysis:

$$\begin{aligned} (\mathcal{M}_{++-}^{1 \text{ loop}})_{\pi^+\pi^-} &= i 2a_{+-}(s_1) v_{\pm}(s_1) R^+(\langle s_1 \rangle) + K'(s_1) \\ &+ (s_1 \leftrightarrow s_2) \end{aligned}$$

where $K'(s_1)$ is polynomial in s_1 (remember that in this case π^- is the odd pion). Then we find:

$$\begin{aligned} \mathcal{M}_{+00}^{\text{fig.3.5}} &= -4 a_x(s_3) a_{+-}(\langle s_3 \rangle) v_{\pm}(\langle s_3 \rangle) R^+(\langle \langle s_3 \rangle \rangle) v_{\pm}(s_3) \\ &+ i 2a_x(s_3) v_{\pm}(s_3) K'(\langle s_3 \rangle) + K''(s_3) \end{aligned} \quad (3.12)$$

with $K''(s_3)$ another polynomial and

$$\langle \langle s_i \rangle \rangle = \frac{3s_0^+ + s_i}{4}$$

In the diagrams as the type in Fig. 3.5, an angular integral such as:

$$\frac{1}{2} \int d(\cos \theta) (\mathcal{M}_{++-}^{1 \text{ loop}})_{\pi^+\pi^-} ((p_1 + q_3)^2)$$

always appears. Following Cabibbo-Isidori, we make a further approximation, replacing the integral with $(\mathcal{M}_{++-}^{1 \text{ loop}})_{\pi^+\pi^-}(\langle s_3 \rangle)$. The second term on the right-hand side of Eq. (3.12) is imaginary and $\propto v_{\pm}(s_3)$ so that it can be added with $\text{Im} \mathcal{B}_{+00}^{(a_i)}$ (Eq. (3.11)) to obtain:

$$2a_x(s_3) v_{\pm}(s_3) [R^+(\langle s_3 \rangle) + K'(\langle s_3 \rangle)]$$

Because of $K'(s_3)$ and $K''(s_3)$ have a polynomial form, we can set $K' = K'' = 0$. According to this prescription, the 2-loops diagrams contribute only to the real part of the amplitude. The other diagrams in Fig. 3.4 can be treated in the same manner, so we find:

$$\begin{aligned} \text{Re } \mathcal{B}_{+00}^{(a_i^2)} = & -2 a_x(s_3) [v_{00}(s_3) a_{00}(s_3) R^+(\langle s_3 \rangle) + R^0(s_3) a_x(s_3) v_{00}(s_3) \\ & + a_{++}(\langle s_3 \rangle) v_{\pm}(\langle s_3 \rangle) R^+(\langle s_3 \rangle) + 2 a_{+-}(\langle s_3 \rangle) v_{\pm}(\langle s_3 \rangle) R^+(\langle s_3 \rangle) \\ & + a_x(\langle s_3 \rangle) v_{00}(\langle s_3 \rangle) R^0(\langle s_3 \rangle)] \end{aligned}$$

As anticipated, the diagrams in Fig. 3.6 give a real contribution, $\delta \text{Re } \mathcal{A}_{+00} \propto v_i(s)$ that cannot be included in R^0 and R^+ (square-root behaviour at the $\pi^0\pi^0$, $\pi^+\pi^0$ thresholds):

$$\begin{aligned} \delta \text{Re } \mathcal{A}_{+00}^{(a_i^2)} = & -4 a_{00}(s_3) a_{+0}(\langle s_3 \rangle) R^+(\langle s_3 \rangle) v_{+0}(\langle s_3 \rangle) v_{00}(s_3) \\ & -2 a_{+0}(s_1) [2 a_{+0}(\langle s_1 \rangle) v_{+0}(\langle s_1 \rangle) R^0(\langle s_1 \rangle) \\ & + a_{00}(\langle s_1 \rangle) v_{00}(\langle s_1 \rangle) R^0(\langle s_1 \rangle) \\ & + 2 a_x(\langle s_1 \rangle) v_{\pm}(\langle s_1 \rangle) R^+(\langle s_1 \rangle) + (s_1 \leftrightarrow s_2)] \end{aligned}$$

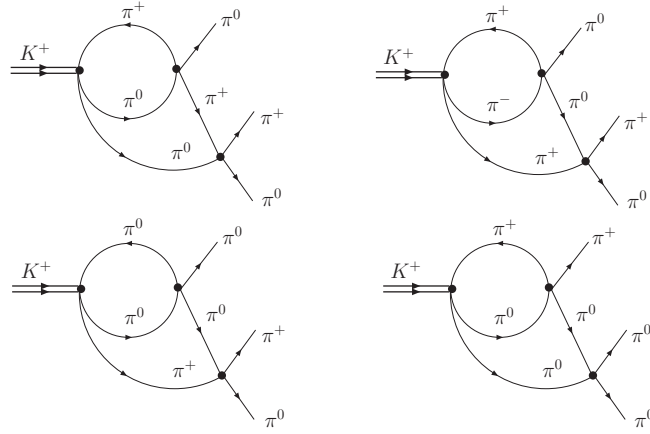


Figure 3.6: The 2-loops diagrams contributing to $\delta \text{Re } \mathcal{A}_{+00} \propto v_i(s)$

We report also the a_i^2 corrections to the real part of $\mathcal{M}_{++-}$ that will be used in 3-loops diagrams (Fig. 3.7, 3.8, 3.9):

$$\delta(\text{Re } \mathcal{D}_{++-}^{(1)})_{Fig. 3.7}^{(a_i^2)} = -4a_x(s_1) a_{+0}(\langle s_1 \rangle) v_{\pm}(\langle s_1 \rangle) R^0(\langle \langle s_1 \rangle \rangle) \quad (3.13)$$

$$\begin{aligned} \delta(\text{Re } \mathcal{C}_{++-})_{Fig. 3.8}^{(a_i^2)} &= -a_{++}(s_3) v_{\pm}(s_3) [2a_x(\langle s_3 \rangle) v_{\pm}(\langle s_3 \rangle) R^0(\langle s_3 \rangle) \\ &+ 4a_{+-}(\langle s_3 \rangle) v_{\pm}(\langle s_3 \rangle) R^+(\langle \langle s_3 \rangle \rangle)] \end{aligned} \quad (3.14)$$

$$\begin{aligned} \delta(\text{Re } \mathcal{C}_{++-})_{Fig. 3.9}^{(a_i^2)} &= -2a_{++}(s_1) v_{\pm}(s_1) [v_{\pm}(\langle s_1 \rangle) R^+(\langle s_1 \rangle) \\ &+ 2a_{+-}(\langle s_1 \rangle) v_{\pm}(\langle s_1 \rangle) R^+(\langle \langle s_1 \rangle \rangle) \\ &+ a_x(\langle s_1 \rangle) v_{00}(\langle s_1 \rangle) R^0(\langle s_1 \rangle)] + (s_1 \leftrightarrow s_2) \end{aligned} \quad (3.15)$$

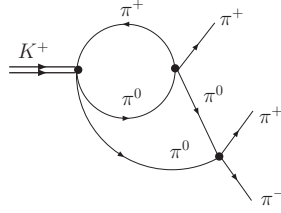


Figure 3.7: The 2-loops diagram that contribute to $(\delta \text{Re } \mathcal{D}_{++-}^{(1)})^{(a_i^2)}$

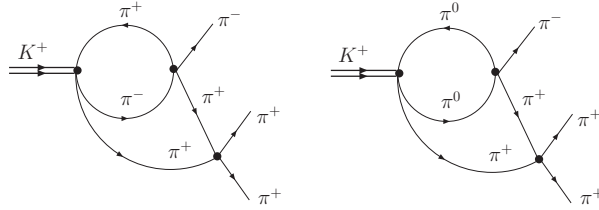
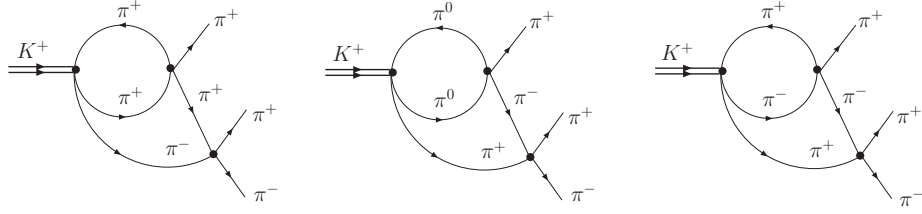


Figure 3.8: The 2-loops diagram that contribute to $(\delta \text{Re } \mathcal{C}_{++-})^{(a_i^2)}$

Figure 3.9: The 2-loops diagram that contribute to $(\delta \text{Re } \mathcal{C}_{++-})^{(a_i^2)}$

At this point, we are able to give the general strategy to expand $|\mathcal{M}_{+00}|^2$ to a given power in a_i . In fact, we have seen that 1-loop diagrams contribute to the imaginary part of the amplitude while 2-loops give real terms. Then, we can assert a priori that 3-loops diagrams can contribute only to the imaginary part. Making use of Eq. (3.3), we can find at which order we have to expand $\mathcal{B}_{+00}$ and $\text{Im}\mathcal{A}_{+00}$ to obtain $|\mathcal{M}_{+00}|^2$ to a given approximation: for example, if we want $|\mathcal{M}_{+00}|^2$ at $\mathcal{O}(a_i^2)$, we need $\text{Im}\mathcal{A}_{+00}$ at the a_i order and $\mathcal{B}_{+00}$ up to a_i^2 . But we know from the previous discussion that $(\text{Im}\mathcal{B}_{+00})^{a_i^2}$ does not exist and then the only a_i^2 term we need is $(\text{Re}\mathcal{B}_{+00})^{a_i^2}$. In table 3.1 are shown the approximations needed for $\text{Im}\mathcal{A}_{+00}$ and $\mathcal{B}_{+00}$ to obtain $|\mathcal{M}_{+00}|^2$ at $\mathcal{O}(a_i^2)$ (left) and at $\mathcal{O}(a_i^3)$ (right). The latter approximation requires the evaluation of 3-loops diagrams that will be calculated in the next chapter.

$ \mathcal{M}_{+00} ^2$ at $\mathcal{O}(a_i^2)$	Powers needed		
	a_i	a_i^2	a_i^3
$\text{Im}\mathcal{A}_{+00}$	•		
$\text{Im}\mathcal{B}_{+00}$	•	-	
$\text{Re}\mathcal{B}_{+00}$	-	•	

(a) The approximation needed for $\mathcal{A}_{+00}$ and $\mathcal{B}_{+00}$ to obtain $|\mathcal{M}_{+00}|^2$ at $\mathcal{O}(a_i^2)$

$ \mathcal{M}_{+00} ^2$ at $\mathcal{O}(a_i^3)$	Powers needed		
	a_i	a_i^2	a_i^3
$\text{Im}\mathcal{A}_{+00}$	•	-	
$\text{Im}\mathcal{B}_{+00}$	•	-	•
$\text{Re}\mathcal{B}_{+00}$	-	•	-

(b) The approximation needed for $\mathcal{A}_{+00}$ and $\mathcal{B}_{+00}$ to obtain $|\mathcal{M}_{+00}|^2$ at $\mathcal{O}(a_i^3)$ Table 3.1: Approximations for $|\mathcal{M}_{+00}|^2$: the dot (•) indicates that the term is needed and exists; the line (-) denotes that the term is needed but is zero.

The $\sqrt{s_3} \equiv M_{\pi^0\pi^0}$ distribution, plotted using the $\mathcal{O}(a_i^2)$ approximation, is shown in Fig. 3.10.

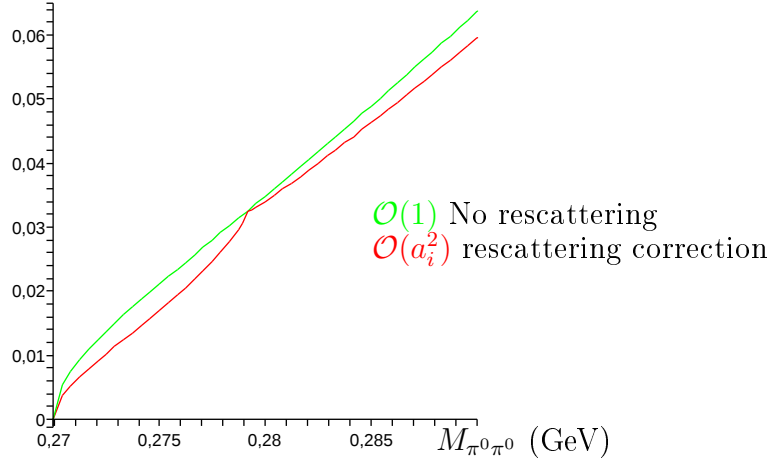


Figure 3.10: The figure shows the $M_{\pi^0\pi^0}$ invariant mass distribution with (*red*) and without (*green*) the $\mathcal{O}(a_i^2)$ re-scattering correction

3.3 $\pi\pi$ rescattering in $K_L^0 \rightarrow 3\pi^0$

Similarly to the $K^+ \rightarrow \pi^+\pi^0\pi^0$ system, the decomposition of the amplitude $\mathcal{M}_{000}$ for the $K_L^0 \rightarrow 3\pi^0$ decay is:

$$\mathcal{M}_{000} = \mathcal{A}_{000} + \sum_{i=1}^3 \mathcal{B}_{000}^{(i)} v_{\pm}(s_i) [\theta(s_i - 4m_{\pi^+}^2) + i\theta(4m_{\pi^+}^2 - s_i)]$$

The tree level amplitudes R_L^0 and R_L^+ for $K_L^0 \rightarrow 3\pi^0$ and $K_L^0 \rightarrow \pi^+\pi^-\pi^0$ decays respectively, have polynomial form and are parametrized as:

$$\text{Re } \mathcal{A}_{000}(s_1, s_2, s_3) \equiv R_L^0(s_1, s_2, s_3) = A_L^0 \left[1 + \frac{h}{2} \cdot \sum_i^3 \frac{(s_i - s_{0_L}^0)^2}{3m_{\pi^+}^4} \right]$$

$$R_L^+(s_3) = A_L^+ \left[1 + g_L^+ \frac{(s_3 - s_{0_L}^+)}{2m_{\pi^+}^2} + \tilde{h}_L^+ \frac{(s_3 - s_{0_L}^+)^2}{2m_{\pi^+}^4} \right]$$

$s_{0_L}^+$, $s_{0_L}^0$ are the centers of the Dalitz plots defined as:

$$s_{0_L}^0 = \frac{1}{3} \sum_{i=1}^3 s_i = \frac{1}{3} (M_{K^0}^2 + 3m_{\pi^0}^2 + m_{\pi^+}^2)$$

$$s_{0_L}^+ = \frac{1}{3} \sum_{i=1}^3 s_i = \frac{1}{3} (M_{K^0}^2 + 2m_{\pi^+}^2 + m_{\pi^0}^2)$$

The parameters h and $g_L^+, \tilde{h}_L^+$ are the slopes of $K_L^0 \rightarrow 3\pi^0$ and $K_L^0 \rightarrow \pi^+\pi^-\pi^0$ decays, respectively. A_L^0 and A_L^+ are the isospin amplitudes. Exact isospin symmetry predicts the ratio (see Appendix A):

$$\frac{A_L^+}{A_L^0} = \frac{1}{3}$$

The calculation of the diagrams at 1-loop order (Fig. 3.11) follows the same steps given by in the K^+ decay and it gives:

$$\begin{aligned} \text{Im } \mathcal{B}_{000}^{(i)} &= 2a_x(s_i) R_L^+(s_i) \\ \text{Im } \mathcal{A}_{000} &= \sum_{i=1}^3 a_{00}(s_i) R_L^0(s_i, \langle s_i \rangle, \langle s_i \rangle) v_{00}(s_i) \end{aligned}$$

Moreover, the 2-loop diagrams give (fig . 3.12 and 3.15):

$$\begin{aligned} \text{Re } \mathcal{B}_{000}^{(i)} &= -2 a_x(s_i) v_{00}(s_i) [R_L^0(s_i, \langle s_i \rangle, \langle s_i \rangle) a_x(s_i) + a_{00}(s_i) R_L^+(s_i)] \\ &\quad - 8 a_x(s_i) a_{+0}(\langle s_i \rangle) R_L^+(\langle s_i \rangle) v_{+0}(\langle s_i \rangle) \end{aligned}$$

where $\langle s_i \rangle = \frac{3s_{0L}^0 - s_i}{2}$ and $\langle\langle s_i \rangle\rangle = \frac{3s_{0L}^0 + s_i}{4}$.

The v -expansion of $|\mathcal{M}_{000}|^2$ at $\mathcal{O}(a_i)$ is then:

$$|\mathcal{M}_{000}(s_1, s_2, s_3)|^2 = R_L^0(s_1, s_2, s_3)^2 - 2 R_L^0(s_1, s_2, s_3) \cdot \left[\sum_{i=1}^3 \text{Im } \mathcal{B}_{000}^i v_{\pm}(s_i) \theta(4m_{\pi^+}^2 - s_i) \right] + \mathcal{O}(a_i^2) \quad (3.16)$$

In this approximation $|\mathcal{M}_{000}|^2$ differs from $(R_L^0)^2$ when $s_i < 4m_{\pi^+}^2$ for some i .

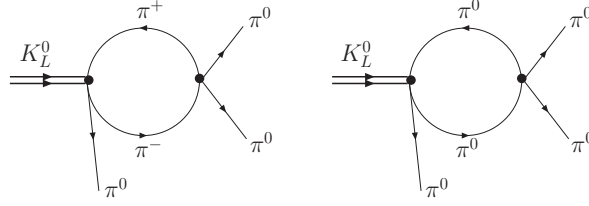
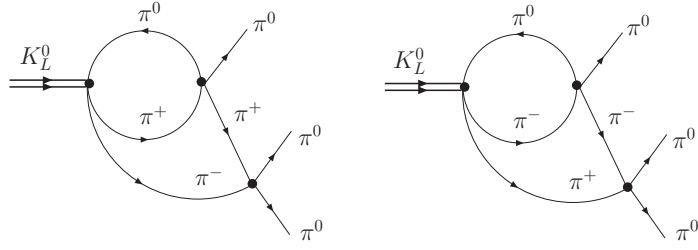
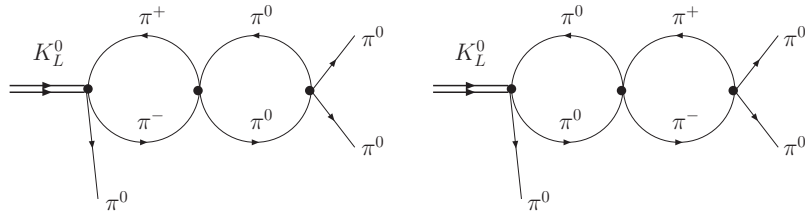
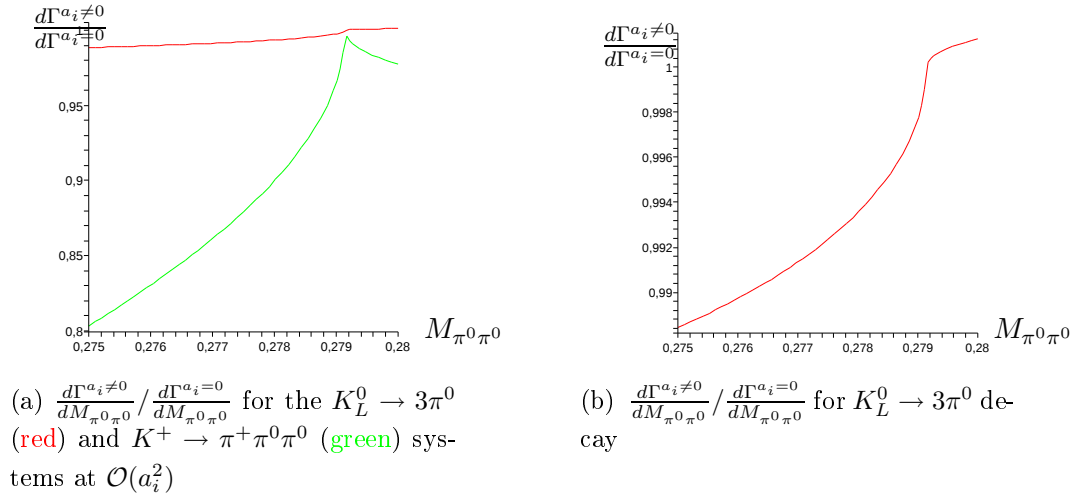


Figure 3.11: 1-loop rescattering diagrams: (left) the charged exchange is the only 1-loop diagram contributing to $|\mathcal{M}_{000}|^2$ at $\mathcal{O}(a_i)$; (right) the neutral exchange diagram

The next-to-leading order is obtained by adding also 2-loops diagrams:

$$\begin{aligned} |\mathcal{M}_{000}(s_1, s_2, s_3)|^2 &= R_L^0(s_1, s_2, s_3)^2 + (\text{Im } \mathcal{A}_{000})^2 + \left[\sum_{i=1}^3 \text{Im } \mathcal{B}_{000}^{(i)} v_{\pm}(s_i) \theta(4m_{\pi^+}^2 - s_i) \right]^2 \\ &+ \left[\sum_{i=1}^3 \text{Im } \mathcal{B}_{000}^i v_{\pm}(s_i) \theta(s_i - 4m_{\pi^+}^2) \right]^2 \\ &- 2 R_L^0(s_1, s_2, s_3) \cdot \left[\sum_{i=1}^3 \text{Im } \mathcal{B}_{000}^{(i)} v_{\pm}(s_i) \theta(4m_{\pi^+}^2 - s_i) \right] \\ &+ \sum_{i=1}^3 2 \left[R_L^0(s_1, s_2, s_3) \text{Re } \mathcal{B}_{000}^{(i)} + \text{Im } \mathcal{A}_{000} \text{Im } \mathcal{B}_{000}^{(i)} \right] \cdot \\ &\cdot v_{\pm}(s_i) \theta(s_i - 4m_{\pi^+}^2) + \mathcal{O}(a_i^3) \end{aligned} \quad (3.17)$$

In Fig. 3.14 is compared the decay rate ratio $\frac{d\Gamma^{a_i \neq 0}}{dM_{\pi^0\pi^0}} / \frac{d\Gamma^{a_i=0}}{dM_{\pi^0\pi^0}}$ for the $K_L^0 \rightarrow 3\pi^0$ and $K^+ \rightarrow \pi^+\pi^0\pi^0$ systems: the cusp effect in the K_L^0 system is strongly suppressed with respect to $K^+ \rightarrow \pi^+\pi^0\pi^0$ decay.

Figure 3.12: *The 2-loops rescattering diagrams that contribute to $\text{Re } \mathcal{B}_{000}^{(i)}$* Figure 3.13: *The 2-loops one-channel rescattering diagrams that contribute to $\text{Re } \mathcal{B}_{000}^i$* Figure 3.14: *Comparison between the $K_L^0 \rightarrow 3\pi^0$ and $K^+ \rightarrow \pi^+\pi^0\pi^0$ cusps*

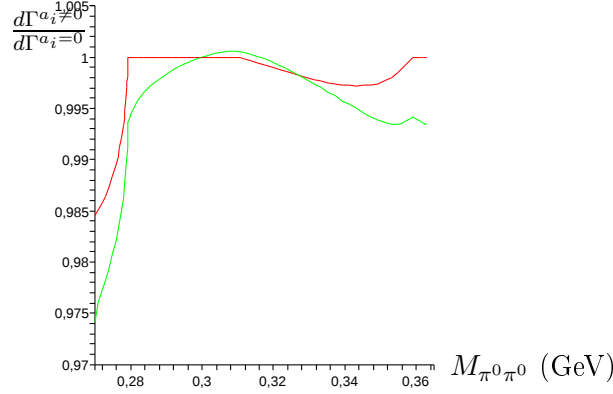


Figure 3.15: $\frac{d\Gamma^{a_i \neq 0}}{dM_{\pi^0\pi^0}} / \frac{d\Gamma^{a_i=0}}{dM_{\pi^0\pi^0}}$ using the $\mathcal{O}(a_i)$ (red) and the $\mathcal{O}(a_i^2)$ (green) approximation for $|\mathcal{M}_{000}|^2$

The cusp has a different importance in the two systems because of differences in their Clebsch-Gordan coefficients (Appendix A): let $\mathcal{M}_{+00}$, $\mathcal{M}_{++-}$ and $\mathcal{M}_{000}$, $\mathcal{M}_{+-0}$ the amplitudes for $K^+ \rightarrow \pi^+\pi^0\pi^0$, $K^+ \rightarrow \pi^+\pi^+\pi^-$ and $K_L^0 \rightarrow 3\pi^0$, $K_L^0 \rightarrow \pi^+\pi^-\pi^0$ respectively; at the leading order (Ref. [7]), the PDG to Cabibbo rate ratio is:

$$\frac{|\mathcal{M}_{+00}|^2}{|\mathcal{M}_{+00}^{(0)}|^2} \begin{cases} 1 & s_3 > 4m_{\pi^\pm}^2 \\ 1 - 4a_x v_\pm \cdot \frac{\mathcal{M}_{++-}^{(0)}}{\mathcal{M}_{+00}^{(0)}} \equiv 1 - 4a_x v_\pm \cdot R_{+00} & s_3 < 4m_{\pi^\pm}^2 \end{cases} \quad (3.18)$$

$$\frac{|\mathcal{M}_{000}|^2}{|\mathcal{M}_{000}^{(0)}|^2} \begin{cases} 1 & s_3 > 4m_{\pi^\pm}^2 \\ 1 - 4a_x v_\pm \cdot \frac{\mathcal{M}_{+-0}^{(0)}}{\mathcal{M}_{000}^{(0)}} \equiv 1 - 4a_x v_\pm \cdot R_{000} & s_3 < 4m_{\pi^\pm}^2 \end{cases} \quad (3.19)$$

$\mathcal{M}^{(0)}$ = old PDG parametrization, $v_\pm = \pi^+\pi^-$ relative velocity in their C.M.

R give the cusp size for the two decays at the leading order and it is dominated by the CG coefficients in the PDG amplitudes (Eq. (3.5), (3.6)). Thus, in $K_L^0 \rightarrow 3\pi^0$ the threshold effect is reduced by CG coefficients alone to 1/6 with respect to K^+ decays. More precisely, by calculating R_{+00} and R_{000} at the threshold and by using the PDG slope values, we have:

$$\frac{R_{+00}}{R_{000}} \sim 13$$

i.e. the cusp “visibility” is about 13 times higher in $K^+ \rightarrow \pi^+\pi^0\pi^0$ decays than in $K_L^0 \rightarrow 3\pi^0$ decays. As a consequence, the cusp is not visible in the data distribution. However, in the data to prediction ratio there is an evidence for a change of slope near the expected cusp point (Fig. 3.16):

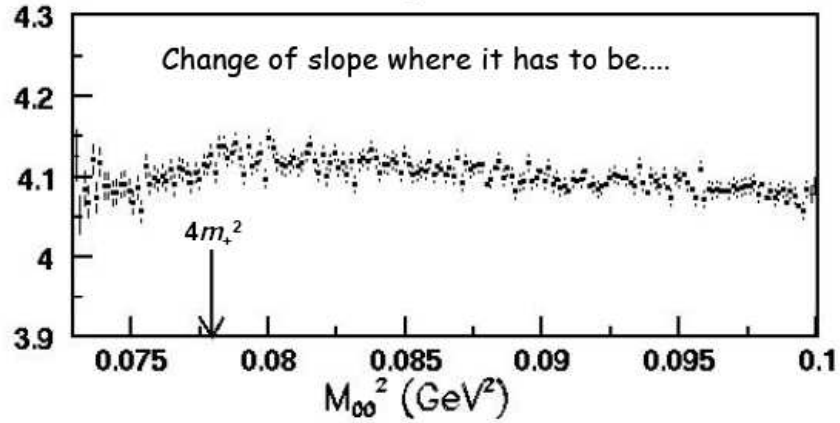


Figure 3.16: *Ratio data prediction of the $M_{\pi^0\pi^0}^2$ distribution. The point where the ratio changes slope is indicated by the arrow: this is at the expected cusp point $M_{\pi^0\pi^0}^2 = 4m_{\pi^\pm}^2$ (taken from Ref. [32])*

Chapter 4

3-loops diagrams

In this chapter we give a detailed calculation of 3-loops diagrams. These diagrams contribute only to $\text{Im}\mathcal{A}_{+00}$ and $\text{Im}\mathcal{B}_{+00}$ at $\mathcal{O}(a_i^3)$ (there is no contribution to the real part because of the redefinition of R^0 and R^+). The aim of this calculation is to expand $|\mathcal{M}_{+00}|^2$ up to a_i^3 terms: according to table 3.1, we see that $(\text{Im}\mathcal{B}_{+00})^{(a_i^3)}$ is the only term needed, then we have to isolate all the diagrams giving this contribution.

4.1 Explicit calculations

There are four topologically distinct 3-loops diagrams (Fig. 4.1), obtained by adding a rescattering vertex at the 2-loops diagrams in all possible ways:

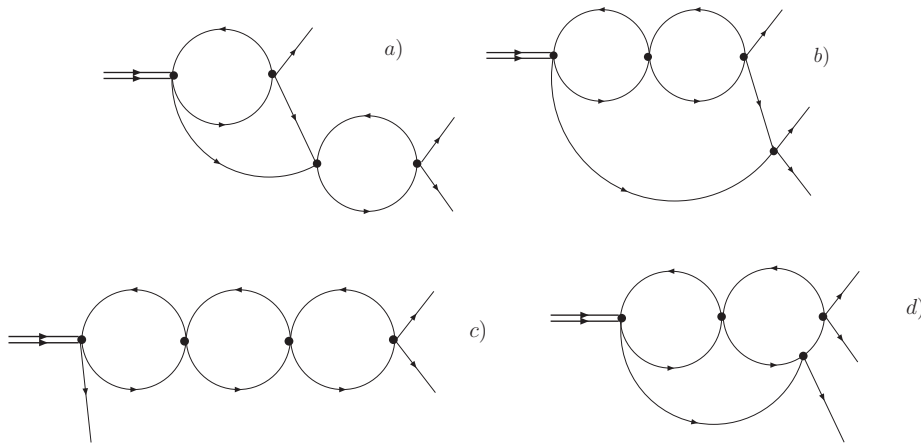


Figure 4.1: *The 3-loops topologies*

- Type a)

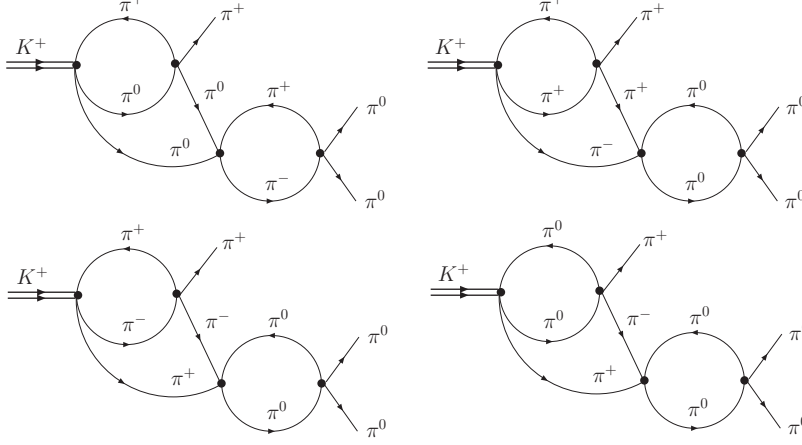


Figure 4.2: The 3-loops diagrams of type a) that give a contribution to $\text{Im } \mathcal{B}_{+00}^{(a^3)}$

These diagrams are obtained from the graphs in Fig. 3.4 (upper row) by adding a vertex in a same rescattering channel (Fig. 4.2):

$$\begin{aligned}
 (\text{Im } \mathcal{B}_{+00})_{\text{type a)}} &= \frac{\pi}{4} v_{\pm}(s_3) \text{Re } \mathcal{A}_x \cdot (\text{Re } \mathcal{D}_{++-}^{(1)}) a_i^2(s_3) + \frac{\pi}{8} v_{00}(s_3) \text{Re } \mathcal{A}_{00} \cdot \\
 &\quad \cdot (\text{Re } \mathcal{B}_{+00}) a_i^2(s_3) \\
 &= -2 a_x(s_3) \cdot [4 a_x(s_3) a_{+0}(\langle s_3 \rangle) v_{0+}(\langle s_3 \rangle) R^0(\langle \langle s_3 \rangle \rangle)] \\
 &\quad - 2 a_{00}(s_3) v_{00}(s_3) a_x(s_3) \cdot [a_{++}(\langle s_3 \rangle) v_{\pm}(\langle s_3 \rangle) R^+(\langle s_3 \rangle) \\
 &\quad + 2 a_{+-}(\langle s_3 \rangle) v_{\pm}(\langle s_3 \rangle) R^+(\langle \langle s_3 \rangle \rangle) \\
 &\quad + a_x(\langle s_3 \rangle) v_{00}(\langle s_3 \rangle) \cdot R^+(\langle s_3 \rangle)]
 \end{aligned}$$

where we used the results given in Eq. (3.13).

- Type b)

These diagrams arise from those on the lower row in Fig. 3.4 when we insert a vertex in a different channel (fig.4.3). In this case, the loop integration (an angular average over the angle between the cut pionic couple) is not trivial. As in the 2-loops case, we adopt a simplification. We begin calculating some of 2-loops graphs of the $K^+ \rightarrow \pi^+ \pi^+ \pi^-$ decay, necessary to evaluate the type b) diagrams (Fig. 4.4). Their sum, denoted as $A^+(s_1, s_2, s_3)$, gives:

$$\begin{aligned}
A^+(s_1, s_2, s_3) = & -a_{++}^2(s_3) v_{\pm}^2(s_3) R^+(s_3) - \{ 4 a_{+-}^2(s_1) v_{\pm}^2(s_1) R^+(\langle s_1 \rangle) \\
& - 2 v_{00}^2(s_1) v_{\pm}^2(s_1) \cdot [a_x^2(s_1) R^+(\langle s_1 \rangle) + a_{+-}(s_1) a_x(s_1) R^0(s_1)] \\
& - v_{00}^2(s_1) a_{00}(s_1) a_x(s_1) R^0(s_1)] + (s_1 \leftrightarrow s_2) \}
\end{aligned}$$

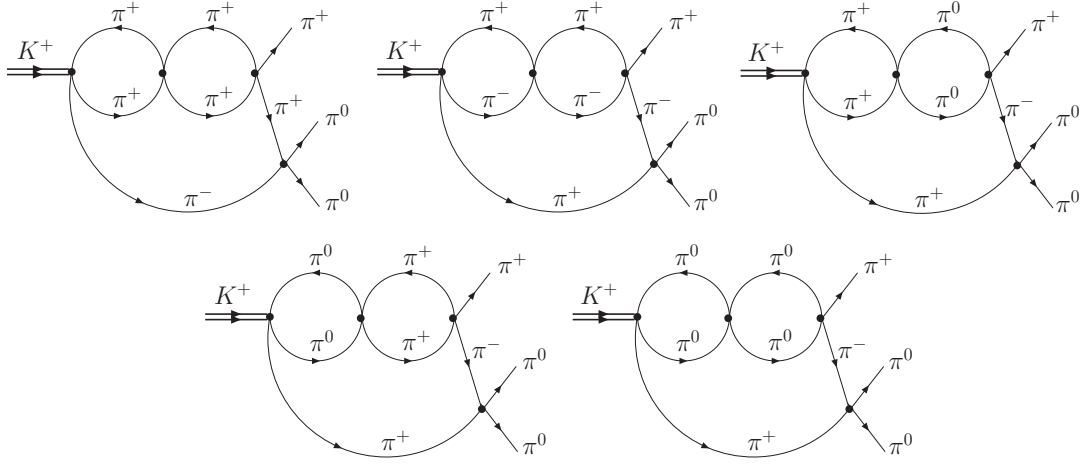


Figure 4.3: The 3-loops diagrams of type b) contributing to $\text{Im } \mathcal{B}_{+00}^{(a^3)}$

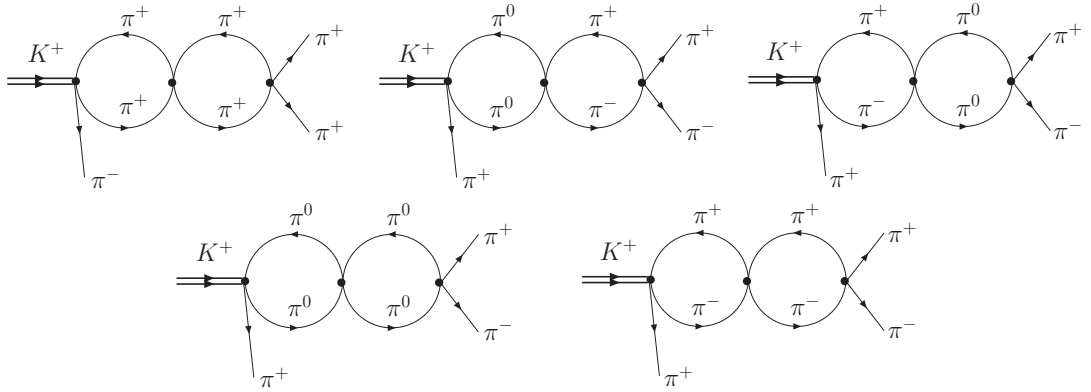


Figure 4.4: The rescattering diagrams of $K^+ \rightarrow \pi^+ \pi^+ \pi^-$ decay occurring in the calculus of the type b) diagrams

Type b) diagrams are evaluated by cutting the last vertex, multiplying $2 a_x(s_3) \cdot v_{\pm}(s_3)$ by $A^+(s_1, s_2, s_3)$ and integrating over the cut $\pi^+ \pi^-$ phase space. The integration can be approximated replacing the integral by $A^+(\langle s_3 \rangle, \langle s_3 \rangle, \langle s_3 \rangle)$.

Finally, we have:

$$\begin{aligned}
(\text{Im } \mathcal{B}_{+00})_{\text{type b}} = & -2 a_x(s_3) \cdot [4 a_{\pm}^2(\langle s_3 \rangle) v_{\pm}^2(\langle s_3 \rangle) R^+(\langle s_3 \rangle) \\
& + a_{++}^2(\langle s_3 \rangle) v_{\pm}^2(\langle s_3 \rangle) R^+(\langle s_3 \rangle) \\
& + 2 v_{\pm}(\langle s_3 \rangle) v_{00}(\langle s_3 \rangle) \cdot (a_{\pm}(\langle s_3 \rangle) a_x(\langle s_3 \rangle) R^0(\langle s_3 \rangle) \\
& + a_x^2(\langle s_3 \rangle) R^+(\langle s_3 \rangle)) + a_x(\langle s_3 \rangle) a_{00}(\langle s_3 \rangle) v_{00}^2(\langle s_3 \rangle) R^0(\langle s_3 \rangle)]
\end{aligned}$$

• Type c)

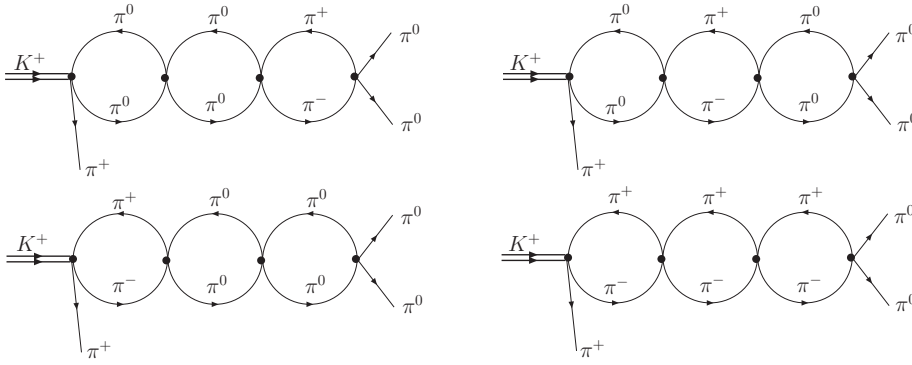


Figure 4.5: The 3-loops diagrams of type c) contributing to $\text{Im } \mathcal{B}_{+00}^{(a^3)}$

Type c) graphs have all the rescattering vertices in the same s_3 channel. The calculus is straightforward and it gives:

$$\begin{aligned}
(\text{Im } \mathcal{B}_{+00})_{\text{type c}} = & -2 a_x(s_3) \cdot [2 a_x(s_3) a_{00}(s_3) v_{00}^2(s_3) R^0(s_3) \\
& + a_{00}^2(s_3) v_{00}^2(s_3) R^+(\langle s_3 \rangle) \\
& + 4 a_{\pm}^2(s_3) v_{\pm}^2(s_3) R^+(\langle s_3 \rangle)]
\end{aligned}$$

- Type d)

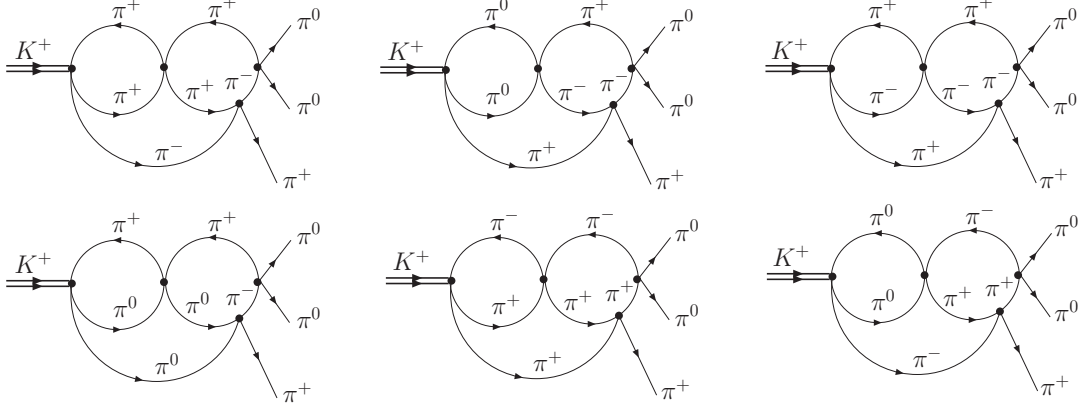


Figure 4.6: The 3-loops diagrams of type d) contributing to $\text{Im } \mathcal{B}_{+00}^{(a^3)}$

In type d) diagrams we have three rescattering vertices in three different channels. As in 2-loops type b) graphs, we simplify the average over the couple angle by making the substitution $s_i \leftrightarrow \langle s_i \rangle$:

$$\begin{aligned}
 (\text{Im } \mathcal{B}_{+00})_{\text{type d)}} &= 2 a_x(s_3) \cdot [\delta \text{Re } \mathcal{C}_{++-}^{a^2}(\langle s_3 \rangle)_{\pi^+\pi^-} + \delta \text{Re } \mathcal{C}_{++-}^{a^2}(\langle s_3 \rangle)_{\pi^+\pi^+} \\
 &\quad + \text{Re } \mathcal{D}_{++-}^{(1) a^2}(\langle s_3 \rangle) v_{00}(\langle s_3 \rangle)] \\
 &= -2 a_x(s_3) \{ 2 a_{\pm}(\langle s_3 \rangle) v_{\pm}(\langle s_3 \rangle) \cdot [a_{++}(\langle\langle s_3 \rangle\rangle) v_{\pm}(\langle\langle s_3 \rangle\rangle) \cdot \\
 &\quad \cdot R^+(\langle\langle s_3 \rangle\rangle) + 2 a_{\pm}(\langle\langle s_3 \rangle\rangle) \cdot v_{\pm}(\langle\langle s_3 \rangle\rangle) R^+(\langle\langle\langle s_3 \rangle\rangle\rangle) \\
 &\quad + a_x(\langle\langle s_3 \rangle\rangle) v_{00}(\langle\langle s_3 \rangle\rangle) R^0(\langle\langle s_3 \rangle\rangle)] \\
 &\quad + a_{++}(\langle s_3 \rangle) v_{\pm}(\langle s_3 \rangle) \cdot [2 a_x(\langle\langle s_3 \rangle\rangle) v_{00}(\langle\langle s_3 \rangle\rangle) R^0(\langle\langle s_3 \rangle\rangle) \\
 &\quad + 4 a_{\pm}(\langle\langle s_3 \rangle\rangle) v_{\pm}(\langle\langle s_3 \rangle\rangle) R^+(\langle\langle\langle s_3 \rangle\rangle\rangle)] \\
 &\quad + 4 a_x(\langle s_3 \rangle) a_{+0}(\langle\langle s_3 \rangle\rangle) v_{+0}(\langle\langle\langle s_3 \rangle\rangle\rangle) R^0(\langle\langle\langle s_3 \rangle\rangle\rangle) \}
 \end{aligned}$$

where:

$$\langle\langle s_i \rangle\rangle \equiv \frac{3 s_0^+ + s_i}{8}$$

Again, we made use of Eq. (3.13)-(3.15) for $(\delta \text{Re } \mathcal{C}_{++-}^{a^2})_{\pi^+\pi^-}$, $(\delta \text{Re } \mathcal{C}_{++-}^{a^2})_{\pi^+\pi^+}$ and $\text{Re } \mathcal{D}_{++-}^{(1) a^2}$.

Finally, we have $\text{Im } \mathcal{B}_{+00}$ at a_i^3 order:

$$\begin{aligned}
\text{Im } \mathcal{B}_{+00}^{a^3} &= (\text{Im } \mathcal{B}_{+00})_a + (\text{Im } \mathcal{B}_{+00})_b + (\text{Im } \mathcal{B}_{+00})_c + (\text{Im } \mathcal{B}_{+00})_d \\
&= -2 a_x(s_3) \cdot [4 a_x(s_3) a_{+0}(\langle s_3 \rangle) v_{0+}(\langle s_3 \rangle) R^0(\langle\langle s_3 \rangle\rangle)] \\
&\quad -2 a_{00}(s_3) v_{00}(s_3) a_x(s_3) \cdot [a_{++}(\langle s_3 \rangle) v_{\pm}(\langle s_3 \rangle) R^+(\langle s_3 \rangle) \\
&\quad + 2 a_{+-}(\langle s_3 \rangle) v_{\pm}(\langle s_3 \rangle) R^+(\langle\langle s_3 \rangle\rangle) + a_x(\langle s_3 \rangle) v_{00}(\langle s_3 \rangle) R^+(\langle s_3 \rangle)] \\
&\quad -2 a_x(s_3) \cdot [4 a_{\pm}^2(\langle s_3 \rangle) v_{\pm}^2(\langle s_3 \rangle) R^+(\langle\langle s_3 \rangle\rangle) + a_{++}^2(\langle s_3 \rangle) v_{\pm}^2(\langle s_3 \rangle) \cdot \\
&\quad \cdot R^+(\langle s_3 \rangle) + 2 v_{\pm}(\langle s_3 \rangle) v_{00}(\langle s_3 \rangle) \cdot (a_{\pm}(\langle s_3 \rangle) a_x(\langle s_3 \rangle) R^0(\langle s_3 \rangle) \\
&\quad + a_x^2(\langle s_3 \rangle) R^+(\langle\langle s_3 \rangle\rangle)) + a_x(\langle s_3 \rangle) a_{00}(\langle s_3 \rangle) v_{00}^2(\langle s_3 \rangle) R^0(\langle s_3 \rangle)] \\
&\quad -2 a_x(s_3) \cdot [2 a_x(s_3) a_{00}(s_3) v_{00}^2(s_3) \cdot R^0(s_3) \\
&\quad + a_{00}^2(s_3) v_{00}^2(s_3) R^+(\langle s_3 \rangle) + 4 a_{\pm}^2(s_3) v_{\pm}^2(s_3) R^+(\langle s_3 \rangle)] \\
&\quad -2 a_x \{ 2 a_{\pm}(\langle s_3 \rangle) v_{\pm}(\langle s_3 \rangle) \cdot [a_{++}(\langle\langle s_3 \rangle\rangle) v_{\pm}(\langle\langle s_3 \rangle\rangle) R^+(\langle\langle s_3 \rangle\rangle) \\
&\quad + 2 a_{\pm}(\langle\langle s_3 \rangle\rangle) \cdot v_{\pm}(\langle\langle s_3 \rangle\rangle) R^+(\langle\langle s_3 \rangle\rangle) + a_x(\langle\langle s_3 \rangle\rangle) v_{00}(\langle\langle s_3 \rangle\rangle) \cdot \\
&\quad \cdot R^0(\langle\langle s_3 \rangle\rangle)] + a_{++}(\langle s_3 \rangle) v_{\pm}(\langle s_3 \rangle) \cdot [2 a_x(\langle\langle s_3 \rangle\rangle) v_{00}(\langle\langle s_3 \rangle\rangle) \cdot \\
&\quad \cdot R^0(\langle\langle s_3 \rangle\rangle) + 4 a_{\pm}(\langle\langle s_3 \rangle\rangle) v_{\pm}(\langle\langle s_3 \rangle\rangle) R^+(\langle\langle\langle s_3 \rangle\rangle)] \\
&\quad + 4 a_x(\langle s_3 \rangle) a_{+0}(\langle\langle s_3 \rangle\rangle) v_{+0}(\langle\langle s_3 \rangle\rangle) R^0(\langle\langle\langle s_3 \rangle\rangle) \} \tag{4.1}
\end{aligned}$$

The effects of the $\text{Im } \mathcal{B}_{+00}^{a^3}$ correction to the amplitude are shown in Fig. 4.7 and 4.8.

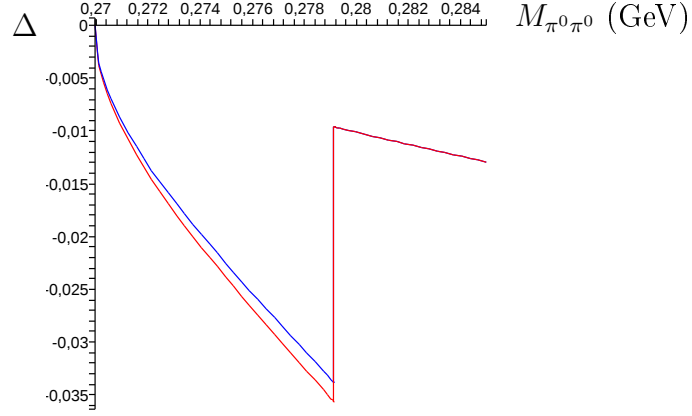


Figure 4.7: Δ_{cusp} at $\mathcal{O}(a_i^2)$ (*red*) and at $\mathcal{O}(a_i^3)$ (*blue*)

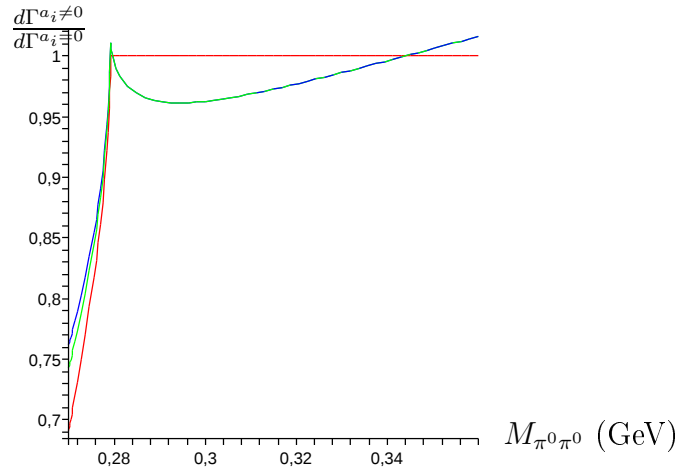


Figure 4.8: Ratio $\frac{d\Gamma^{a_i \neq 0}}{dM_{\pi^0\pi^0}} / \frac{d\Gamma^{a_i = 0}}{dM_{\pi^0\pi^0}}$ at $\mathcal{O}(a_i)$ (*red*), at $\mathcal{O}(a_i^2)$ (*green*) and at $\mathcal{O}(a_i^3)$ (*blue*)

4.2 Estimation of the theoretical error due to the a_i expansion

The Cabibbo-Isidori proposal was to give a systematic expansion of $|\mathcal{M}_{+00}|^2$ in ascending powers of a_i in order to extract the scattering lengths and the slope parameters by a fitting procedure. In this model, the theoretical error afflicts the scattering lengths and slopes and it's due to the discarding of a_i^3 and higher terms in $|\mathcal{M}_{+00}|^2$. Moreover, the radiative corrections and isospin-breaking effects that we did not investigated, are also sources of uncertainty. The theoretical uncertainty should be evaluated by fitting a_0 and a_2 and comparing the central values obtained with and without the correction. Nevertheless, we can give an estimation of the theoretical error by adjusting a_0 , a_2 and the slopes in $\Delta_{cusp}^{(a_i^2)}$ by a some percent with respect to the quoted PDG values in order to reproduce the $\Delta_{cusp}^{(a_i^3)}$ function. Cabibbo-Isidori proposed a naive 5% error on the scattering lengths but we expect that a smaller change on the parameters could simulate the $\mathcal{O}(a_i^3)$ approximation.

In the following, we varied the slope parameters g^0 , $\tilde{h}^0$ and the scattering lengths (they are also the fitted quantities extracted in the analysis of Ref. [10]): any other parameter (for example the slope parameters of $K^+ \rightarrow \pi^+\pi^+\pi^-$ decay) does not produce any appreciable difference when varied: some differences appear only for very large changes (20% or more) but they cannot be imputed to a theoretical error (Fig. 4.9). The value of the parameters used in $\Delta_{cusp}^{(a_i^3)}$ are taken from Ref. [13,31] (table 4.1):

Parameter	Value
a_0	0.221 ± 0.005
a_2	-0.044 ± 0.001
r_0	1.25 ± 0.04
r_2	1.81 ± 0.05
g^0	0.626 ± 0.007
$\tilde{h}^0$	-0.046 ± 0.01
g^+	-0.2154 ± 0.0035
$\tilde{h}^+$	0.0004 ± 0.004

Table 4.1: *The values for the slope parameters and the scattering lengths are taken from PDG and Ref. [31]*

The parameters $\tilde{h}^0, \tilde{h}^+$ are related to the PDG quadratic slope values h^0, h^+ by ¹:

$$\tilde{h}^0 = h^0 - (g^0/2)^2 \quad \tilde{h}^+ = h^+ - (g^+/2)^2 \quad (4.2)$$

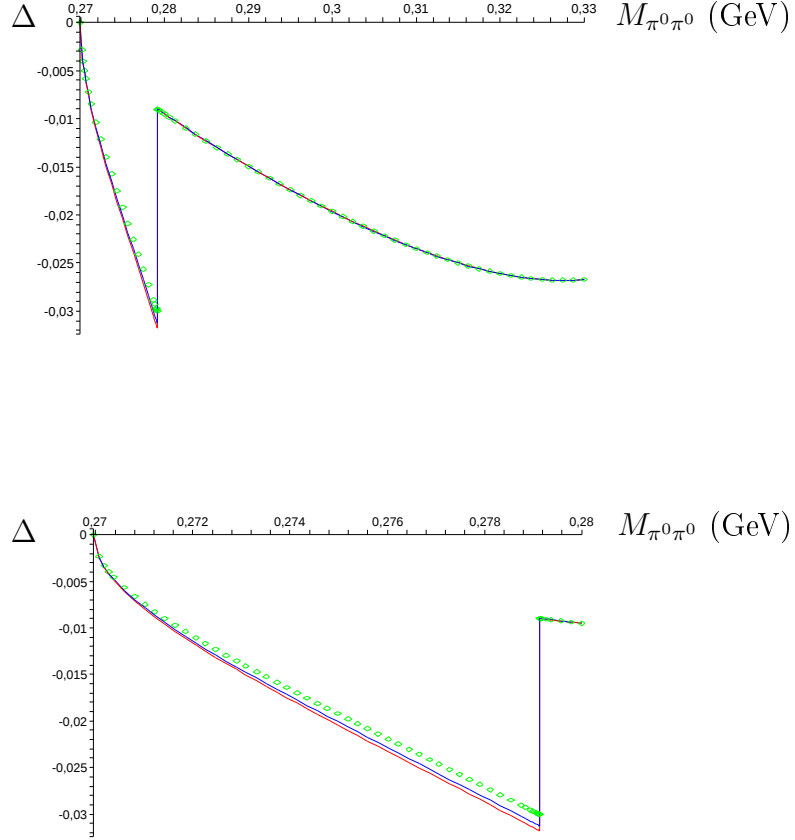


Figure 4.9: The figures show $\Delta_{cusp}^{(a_i^2)}$ with $g^+ = -0.270$ (blue), compared with $\Delta_{cusp}^{(a_i^2)}$ (red) and $\Delta_{cusp}^{(a_i^3)}$ (green), both plotted with values of table 4.1. A change of 25% in g^+ does not produce any appreciable difference from the red curve. The lower figure is an enlargement of the upper plot near the cusp zone

¹In fact, the PDG amplitude is $\sqrt{1 + g^{(+,0)} \frac{(s_3 - s_0)}{m_{\pi^\pm}^2} + h^{(+,0)} \frac{(s_3 - s_0)^2}{m_{\pi^\pm}^4} + \dots}$. Expanding the square root up to the quadratic terms, we obtain Eq. (4.2)

Changing a_0 and a_2 by $\pm 5\%$ from the value reported in table 4.1, we found that $\Delta_{cusp}^{(a_i^2)}$ fits the $\Delta_{cusp}^{(a_i^3)}$ curve quite well inside the cusp zone but it differs in the remaining region (Fig. 4.10). Varying g^0 by the same amount, we obtained a good matching both below and above the cusp (Fig. 4.12).

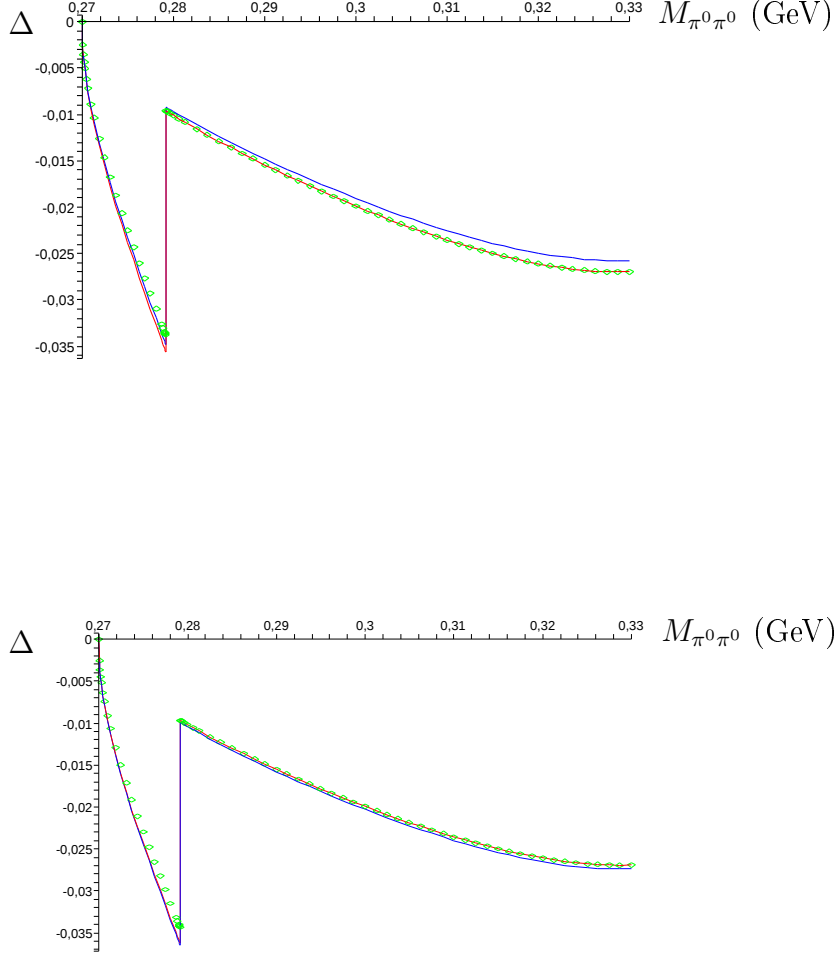


Figure 4.10: The figure shows $\Delta_{cusp}^{(a_i^2)}$ when we have set $a_0 = 0.210$ (blue, upper plot) and $a_2 = -0.046$ (blue, lower plot), compared with $\Delta_{cusp}^{(a_i^2)}$ (red) and $\Delta_{cusp}^{(a_i^3)}$ (green). Changing the scattering lengths by 5% we found a good matching below the cusp but a notable difference above the threshold.

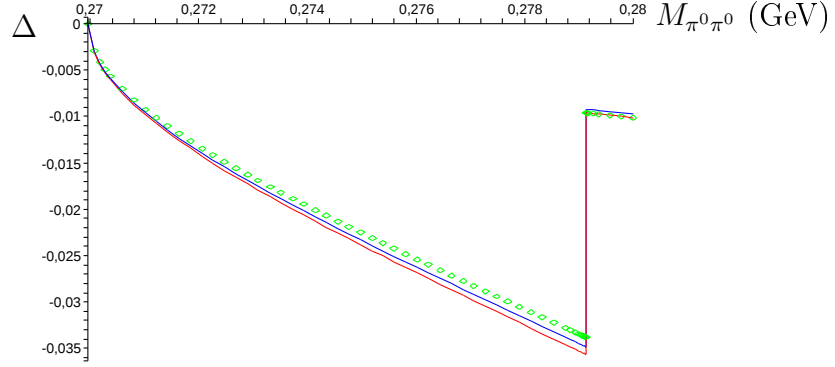


Figure 4.11: *This figure is an enlargement of Fig. 4.10 in the zone below the cusp. We note that a change of 5% in a_0 does not yield to a perfect agreement with $\Delta_{cusp}^{(a_i^3)}$ (green points)*

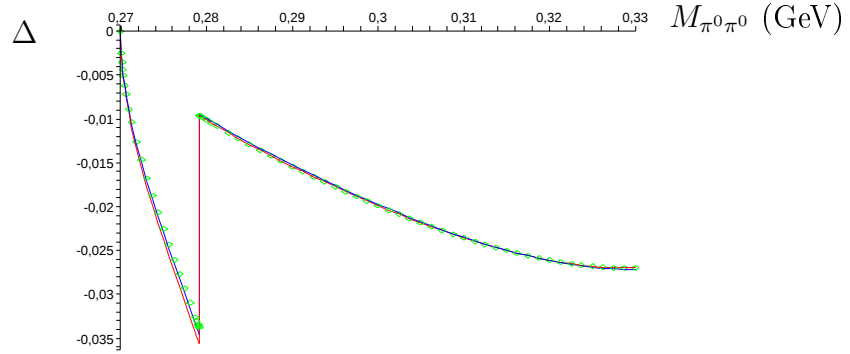


Figure 4.12: *The figure shows $\Delta_{cusp}^{(a_i^2)}$ where we have set $g^0 = 0.670$ (instead of $g_0 = 0.626$) (blue). In this case the agreement with $\Delta_{cusp}^{(a_i^3)}$ (green) is equally good below and above the cusp; in particular these curves match together above the cusp*

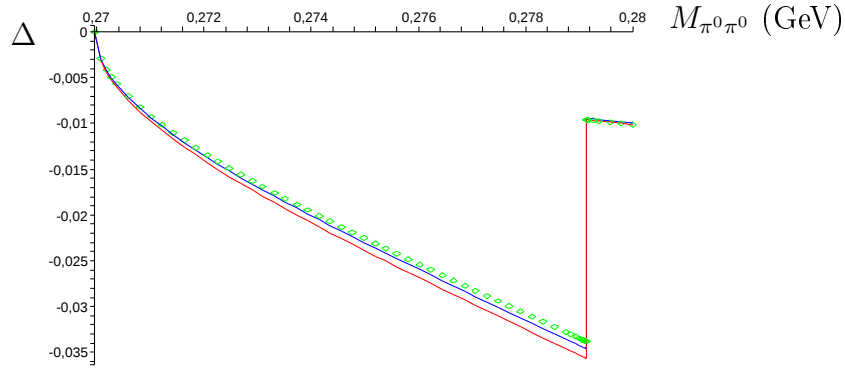


Figure 4.13: *This figure is an enlargement of Fig. 4.12 in the zone below the cusp. The agreement with $\Delta_{cusp}^{(a_i^3)}$ (green) is better than shown in Fig. 4.10 also below the cusp*

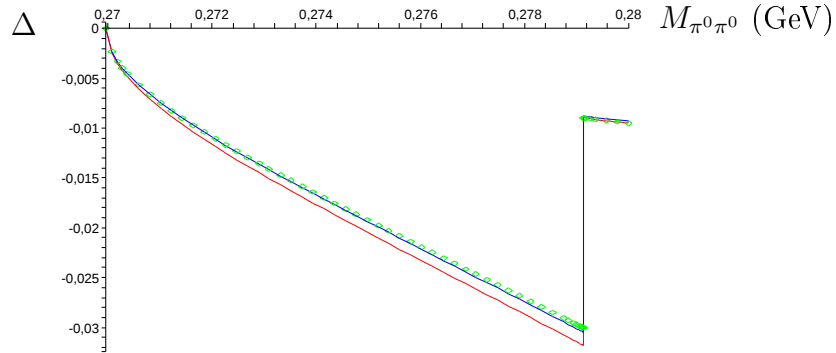


Figure 4.14: *This figure shows $\Delta_{cusp}^{(a_i^2)}$ with g^0 and $\tilde{h}^0$ shifted by 5% (blue). The agreement with $\Delta_{cusp}^{(a_i^3)}$ (green) is slightly better than shown in Fig. 4.12.*

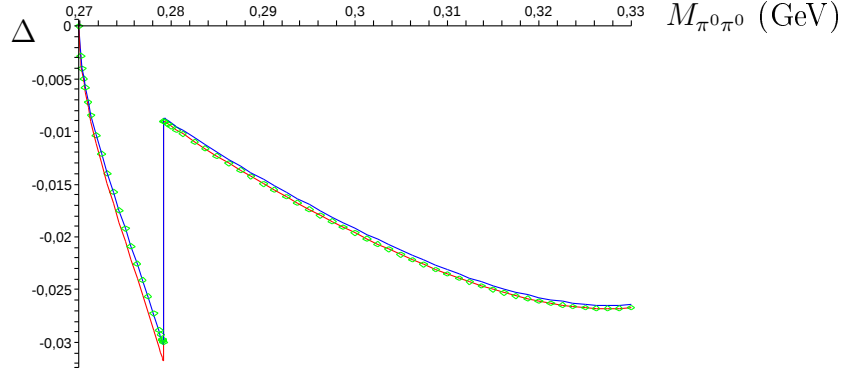


Figure 4.15: This figure shows $\Delta_{cusp}^{(a_i^2)}$ with a_0 shifted by 2% (blue) with respect to values in table 4.1. The blue curve fits $\Delta_{cusp}^{(a_i^3)}$ (green) as in Fig. 4.14, where a_0 was not changed, but it slightly deviates from that above the threshold

From Fig. 4.9 - 4.14, it's clear that the 5% naive estimation on the theoretical error given by Cabibbo-Isidori was pessimistic and it can be reasonably reduced. In fact the error committed neglecting the a^3 and higher terms, translates in a theoretical uncertainty on the slope parameters $\tilde{h}^0$ and g^0 (5%) but on the contrary, it does not give an appreciable error on the scattering lengths. In Fig. 4.15 it's shown $\Delta_{cusp}^{(a_i^2)}$ when a_0 is shifted by 2% (a_2 does not produce any appreciable effect): we observe that $\Delta_{cusp}^{(a_i^2)}$ continues to fit the $\Delta_{cusp}^{(a_i^3)}$ curve below the cusp but this is not reproduced with the same precision above the threshold. In this zone, $\Delta_{cusp}^{(a_i^2)}$ begins to deviate slightly from $\Delta_{cusp}^{(a_i^3)}$ and a further change of the scattering lengths values (for example 3 – 4%) does not give a reasonable matching. In order to correct this difference, we have to change the slope parameters by 20% or more, a very large tuning for a small correction of a_i^3 size.

From this analysis we conclude that the theoretical error on a_i , committed when the a_i^3 and higher terms are discarded from $|\mathcal{M}_{+00}|^2$, does not exceed the 2% level. Effects coming from these terms can be simply absorbed in shift of 5% on the $K^+ \rightarrow \pi^+\pi^0\pi^0$ slope parameters $\tilde{h}^0$ and g^0 .

Chapter 5

Summary and conclusion

We have presented a 3-loops calculation based on the Cabibbo-Isidori approach to the $\pi\pi$ rescattering in $K^+ \rightarrow \pi^+\pi^0\pi^0$ decays, the final state interaction responsible for the cusp effect visible in the experimental $\pi^0\pi^0$ invariant mass distribution (Ref. [10] and [9]). This method gives the decay matrix element $\mathcal{M}_{+00}$ as a power series in the pion S-wave scattering lengths a_0 and a_2 , supposed to be small according to the ChPT prediction: $a_0 = 0.220 \pm 0.005$, $a_2 = -0.0444 \pm 0.0010$ (in m_{π^+} units). In this way the scattering lengths and the slope parameters appear as free parameters to be extracted by studying the $\pi^0\pi^0$ spectrum. The aim of this work was to expand $|\mathcal{M}_{+00}|^2$ up to $\mathcal{O}(a_i^3)$ using the 3-loops diagrams and give an estimate for the theoretical error on the scattering lengths, reducing the 5% error quoted by Cabibbo-Isidori.

As a first step, we have explicit the term proportional to the $\pi^+\pi^-$ velocity, decomposing the amplitude as $\mathcal{M}_{+00} \equiv \mathcal{A}_{+00} + \mathcal{B}_{+00} \cdot v_{\pm}(s_3)$, where $\text{Im}\mathcal{A}_{+00}$ and $\mathcal{B}_{+00}$ are the non-analytic terms arising when two or more pions rescatter in a diagram (they are both regular at the $s_3 = 4m_{\pi^+}^2$ threshold). Squaring $\mathcal{M}_{+00}$, we have noted that the only missing term relevant at a_i^3 order is $\text{Im}\mathcal{B}_{+00}^{a_i^3}$. We have listed all the topologically distinct 3-loops diagrams obtained by adding a rescattering vertex to the 2-loops graphs (in all possible ways) and we have collected those giving a contribution to $\text{Im}\mathcal{B}_{+00}$. Finally, for each of these graphs we have computed the imaginary parts of $\mathcal{B}_{+00}$ by using the Cutkosky rules. This a_i^3 correction acts only below the threshold: above the cusp $|\mathcal{M}_{+00}|^2$ at $\mathcal{O}(a_i^3)$ is equal to $|\mathcal{M}_{+00}|^2$ at $\mathcal{O}(a_i^2)$.

The theoretical error on the scattering lengths was estimated by tuning a_0 , a_2 and the slope parameters in the $M_{\pi^0\pi^0}$ distribution at $\mathcal{O}(a_i^2)$ as long as this curve did not match with that at $\mathcal{O}(a_i^3)$. We have found that a small shift of 5% in the scattering lengths (from the predicted value) produces a good

agreement with the $\mathcal{O}(a_i^3)$ distribution below the cusp but it differs far from the threshold. On the contrary, by changing only the $K^+ \rightarrow \pi^+\pi^0\pi^0$ slope parameters by a 5%, we have obtained a good matching both below and above the cusp zone. This result confirms that the 5% error on the scattering lengths was pessimistic: the theoretical error committed neglecting the a_i^3 and higher terms can be in principle reabsorbed in a shift on the slope parameters. From this analysis we have concluded that the error does not exceed the 2% level. A posteriori checks about the size of this error could be obtained by a complete fit of the data with and without the a_i^3 correction computed here.

Part II

A cusp-like structure in the $\pi^0\pi^0$
invariant mass from $K_L^0 \rightarrow 3\pi^0$
decay in the NA48 experiment

Chapter 6

Experimental results

In this chapter we review the known methods for the extraction of the charge-exchange scattering length $a_x = (a_0 - a_2)/3$ and the slope parameter h . Moreover we list the recent experiments that have measured these quantities.

6.1 Measurements of $a_0 - a_2$

6.1.1 $a_0 - a_2$ from $K^+ \rightarrow \pi^+ \pi^- e^+ \nu$ decays (K_{e4})

The first, naive explanation, that is it possible to measure $\pi\pi$ scattering in such a kaon decay process is that the pions undergo final-state interactions (Fig. 6.1) and they are the only strongly interacting particles in the final state, therefore they ought to be sensitive to this interaction.

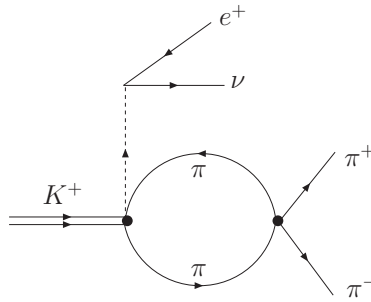


Figure 6.1: $\pi\pi$ scattering in K_{e4} decay

The decay

$$K^+(p) \rightarrow \pi^+(p_1) \pi^-(p_2) e^+(p_e) \nu(p_\nu) \quad (6.1)$$

can most conveniently be treated by using three reference frames, as illustrated in Fig. 6.2: (1) the K^+ rest system (Σ_K), (2) the $\pi^+\pi^-$ rest system ($\Sigma_{\pi\pi}$) and (3) the $e^+\nu$ rest system ($\Sigma_{e\nu}$). The kinematics of the K_{e4} decay are then fully described by five independent¹ variables, introduced by Cabibbo and Maksymowicz (Ref. [33]):

1. $s_\pi = M_{\pi^0\pi^0}^2$ the invariant mass squared of the dipion;
2. $s_e = M_{e\nu}^2$ the invariant mass squared of the dilepton;
3. θ_π the angle of the π^+ in $\Sigma_{\pi\pi}$ with respect to the direction of flight of the dipion in Σ_K ;
4. θ_e the angle of the e^+ in $\Sigma_{e\nu}$ with respect to the direction of flight of the dilepton in Σ_K ;
5. ϕ , the angle between the plane formed by the two pions and the corresponding plane formed by the two leptons.

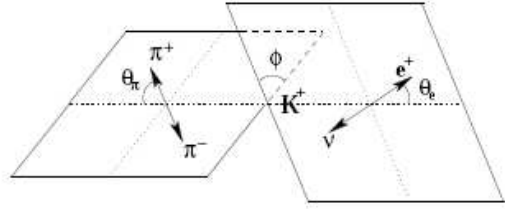


Figure 6.2: *Kinematic quantities used in the analysis of K_{e4} decay*

It is useful for the following discussion to introduce the combinations P , Q and L of the momentum four vectors p_1 , p_2 , p_e and p_ν defined in Eq. (6.1) and two scalar products derived from them

$$P = p_1 + p_2, \quad Q = p_1 - p_2, \quad L = p_e + p_\nu$$

$$Q^2 = 4m_\pi^2 - s_\pi, \quad P \cdot L = \frac{1}{2}(M_K^2 - s_e - s_\pi)$$

¹For a process involving N particles (initial + final), the number of independent variables is $3N - 10$.

The decay matrix element is

$$\mathcal{M}_{K_{e4}} = \frac{G_F}{\sqrt{2}} V_{us}^* \bar{u}(p_\nu) \gamma_\mu (1 - \gamma_5) v(p_{e^+}) (V^\mu - A^\mu)$$

where V_{us}^* is the CKM matrix element for the $s \rightarrow u$ transition.

The form of vector current V^μ and the axial current A^μ is dictated by Lorentz invariance:

$$\begin{aligned} V^\mu &= \frac{1}{M_K^3} (H \epsilon^{\mu\nu\rho\sigma} P_\nu Q_\rho L_\sigma) \\ A^\mu &= \frac{1}{M_K} (F P^\mu + G Q^\mu + R L^\mu) \end{aligned}$$

The form factors F, G, R, H so defined are adimensional and functions of the Cabibbo-Maksymowicz variables. The decay rate follows from the matrix element and neglecting terms proportional to m^2/s_e (for more details see Ref. [34]). A parametrisation for the form factors, introduced by Amorós and Bijmans (Ref. [35]) may be used, which is based on a partial wave expansion in the variable θ_π :

$$\begin{aligned} F &= (f_s + f'_s q^2 + f''_s q^4 + f_e(s_e/4m_\pi^2)) e^{i\delta_0^0(s_\pi)} + \tilde{f}_p \cos\theta_\pi e^{i\delta_1^1(s_\pi)} \\ G &= (g_p + g'_p q^2 + g_e(s_e/4m_\pi^2)) e^{i\delta_1^1(s_\pi)} \\ H &= (h_p + h'_p q^2) e^{i\delta_1^1(s_\pi)} \end{aligned}$$

where q is the pion momentum in $\Sigma_{\pi\pi}$. It yields 10 new dimensionless form factor parameters $f_s, f'_s, f''_s, f_e, \tilde{f}_p, g_p, g'_p, g_e, h_p$ and h'_p which do not depend on any kinematic variables, plus two phase $\pi\pi$ scattering phase shifts $\delta_{\ell=0}^{I=0}$ and $\delta_{\ell=1}^{I=1}$ (s_π -dependent); the phase shifts enter in the rate only in the combination $\delta = \delta_0^0 - \delta_1^1$, so that the physical quantities that can be fitted are the form factors parameters and δ . The kinematic dependence of $\delta(s_\pi)$ was proposed by Schenk (Ref. [36]):

$$\tan(\delta_\ell^I) = \sqrt{1 - \frac{4m_\pi^2}{s_\pi}} q^{2\ell} \{A_\ell^I + B_\ell^I q^2 + \dots\} \times \frac{4m_\pi^2 - s_\ell^I}{s_\pi - s_\ell^I}$$

The parameters A_ℓ^I and B_ℓ^I , etc. can be expressed as a function of only two parameters or subtraction constants, which are identified as the $I = 0$ and $I = 2$ S -wave scattering lengths a_0 and a_2 . For example, the first two coefficients of this expression for the $I = \ell = 0$ case read as follows:

$$\begin{aligned} A_0^0 &= a_0 \\ B_0^0 &= 0.2395 + 0.9237 \Delta a_0 - 3.352 \Delta a_2 + 0.2817 (\Delta a_0)^2 \\ &\quad + 6.335 (\Delta a_2)^2 + 6.074 \Delta a_0 \Delta a_2 + \dots \end{aligned}$$

where $\Delta a_0 = a_0 - 0.220$ and $\Delta a_2 = a_2 + 0.0444$. Although K_{e4} decay allows only $I = 0$ and $I = 1$ contributions, the use of the crossing relations brings in a modest dependence on the $I = 2$ scattering length. It was pointed out by Morgan, Shaw (Ref. [29]) and Colangelo, Gasser and Leutwyler (Ref. [11], [28]) that the possible values of a_2 and a_0 are restricted to a band in the (a_0, a_2) plane, the so-called universal band (Chapter 1); the central curve of this band is given by ²:

$$a_2 = -0.0849 + 0.232 a_0 - 0.0865 a_0^2 [\pm 0.0088]$$

a_2 and a_0 are then related as:

$$\Delta a_2 = 0.236 \Delta a_0 - 0.61 (\Delta a_0)^2 - 9.9 (\Delta a_0)^3 \quad (6.2)$$

Therefore, theoretical arguments suggest that the charge-exchange scattering length $a_x = (a_0 - a_2)/3$ can be extracted from the K_{e4} decays.

Since the low K_{e4} branching ratio ($\sim 10^{-5}$), it was not until 1977, when the Geneva-Saclay experiment (Ref. [37]) gathered about 30,000 events, that a measurement was made of a_0 to 20% accuracy. New data and a new analysis become available with the Experiment 865 (E865) at the Brookhaven Alternating Gradient Synchrotron that collected 400,000 K_{e4} events (Ref. [38]). Recently, a new measurement was performed by NA48/2 collaboration (Ref. [39]) based on a partial sample of more than 670,000 decays. The available experimental results of E865 and NA4/2 are summarized in table 6.1:

Experiment	2-p fit		U.B. (1-p fit)	
	a_0	a_2	a_0	a_2 (from Eq. (6.2))
E865 [38]	0.203 ± 0.033	-0.055 ± 0.023	0.216 ± 0.014	-0.0454 ± 0.003
NA48/2 [39]	0.233 ± 0.017	-0.047 ± 0.012	0.256 ± 0.019	-0.0312 ± 0.013

Table 6.1: *The E865 and the NA48/2 results for the a_0 and a_2 scattering lengths, when they were floated in the fit (2-parameter fit) and when a_2 was constrained by Eq. (6.2) (1-parameter fit)*

²We remember that we are using $m_{\pi^\pm}$ as unit of energy for the scattering lengths.

6.1.2 $a_0 - a_2$ from $\pi^0\pi^0$ decay of the pionic atom $\pi^+\pi^-$

The idea of obtaining the charge-exchange combination a_x from studying the $\pi^+\pi^-$ bound state (pionium), denoted as $A_{2\pi}$, was proposed by Nemenov in 1985 (Ref. [40]). The main characteristics of such atom are (Ref. [41]): binding energy 1.9 KeV; $J^{PC}(1S) = 0^{++}$; Bhorr radius $r_B = 387$ fm. The dominant decay mode for pionium is $A_{2\pi} \rightarrow \pi^0\pi^0$; this channel largely exceeds $\gamma\gamma$:

$$\frac{\Gamma_{\gamma\gamma}}{\Gamma_{\pi^0\pi^0}} \sim 4 \cdot 10^{-3}$$

The decay width is then dominated by the strong reaction $\pi^+\pi^- \rightarrow \pi^0\pi^0$ (Fig. 6.3):

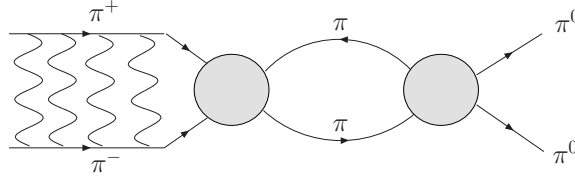


Figure 6.3: $\pi\pi$ scattering in the $A_{2\pi} \rightarrow \pi^0\pi^0$ system: wavy lines are photons binding the $\pi^+\pi^-$ couple; the blobs are the strong interaction vertices

Applying Chiral Perturbation theory methods Gasser et al. (Ref. [42]) have taken into account higher order corrections with respect to the leading and next-to-leading order terms (Ref. [43]) and obtained the $1s$ state lifetime given by:

$$\Gamma_{\pi^0\pi^0} = \frac{16\pi}{9} \alpha^3 \frac{p_{n=1}^*}{m_{\pi^+}} (a_0 - a_2)^2 |\Psi_{n=1}(0)|^2 (1 + \delta_\Gamma)$$

where $\Psi_{n=1}(0)$ is the hydrogen-like wave function for $\pi^+\pi^-$ at $r = 0$,

$$p_{n=1}^* = \sqrt{m_\pi^2 - m_{\pi^0}^2 - 1/4(m_\pi^2 \alpha^2)}$$

α is the fine structure constant and $\delta_\Gamma = (5.8 \pm 1.2) \cdot 10^{-2}$ is a parameter that arise from higher terms corrections. Combined with the theoretical values for a_0 and a_2 (Ref. [31]), $1s$ lifetime is predicted to be:

$$\tau_{1s} = (2.9 \pm 0.1) \cdot 10^{-15} \text{ sec}$$

It should be noted that a $a_0 - a_2$ measurement from pionium decay provides only a determination of $|a_0 - a_2|$.

The DIRAC (Dimeson Relativistic Atomic Complex) experiment at CERN is devoted to the study of the ponium lifetime, in particular to extract the $|a_0 - a_2|$ charge-exchange scattering length with a 5% precision. Under the experimental conditions of the DIRAC experiment, a 24(20) GeV proton beam collides with a nuclear target. The proton-target nuclei interactions may create pions through direct production and through decays of mesons generated by primary interaction. The charged pions, when their relative distance is of the order of few fm, can bind together via Coulomb attraction to form the pionic atom $A_{2\pi}$ in the same nuclear target. Once created, an $A_{2\pi}$ may evolve in three different ways by interacting with the electric field of the target nuclei: it may annihilate in $2\pi^0$ by strong interaction, be (de-)excited to a higher (lower) n-state or break up in $\pi^+\pi^-$ (ionization); the latter interactions are purely electromagnetic. The probabilities of annihilation (P_{anh}), (de-)excitation (P_{dsc}) and breakup (P_{break}) constitute a “complete set” of transformations for the atom, whose evolution at any time is governed by:

$$P_{\text{anh}} + P_{\text{break}} + P_{\text{dsc}} = 1$$

When a break-up occurs a $\pi^+\pi^-$ pair of opposite charge and opposite momentum is produced having very small opening angle (*atomic pairs*). The aim of DIRAC experiment was to measure the probability of ponium breakup by detecting these atomic pairs with an high resolution double arm magnetic spectrometer. By exploiting the “completeness” relationship above, the breakup probability can be linked to ponium lifetime for a particular choice of target. The results for the 1s ponium lifetime and $a_0 - a_2$ from about 6530 dissociated $A_{2\pi}$ atoms, are (Ref. [41]):

$$\tau_{1s} = (2.91_{-0.62}^{+0.49}) \cdot 10^{-15} \text{ sec}$$

$$|a_0 - a_2| = 0.264_{-0.020}^{+0.033}$$

in agreement with the other experiments and with theoretical predictions.

6.1.3 $a_0 - a_2$ from $K^+ \rightarrow \pi^+ \pi^0 \pi^0$

The Cabibbo cusp-effect (Ref. [7, 8]) was observed for the first time in 2005 by the NA48/2 collaboration studying the $\pi^0 \pi^0$ invariant mass spectrum from a partial sample of about $2.7 \cdot 10^7$ events (Ref. [9], [10]). This method for the extraction of the $a_0 - a_2$ scattering length was theoretically discussed in part I so that we give only the experimental result:

$$a_0 - a_2 = 0.268 \pm 0.010_{stat.} \pm 0.004_{syst.} \pm 0.013_{ext.}$$

$$a_2 = -0.041 \pm 0.022_{stat} \pm 0.014_{syst.}$$

This result is in good agreement with the Universal Band constraint (Eq. (6.2)) in fact if one link a_0 to a_2 using the correlation provided by U.B., the scattering lengths become:

$$a_0 = 0.220 \pm 0.006_{stat.} \pm 0.004_{syst.} \pm 0.011_{ext.}$$

$$a_0 - a_2 = 0.264 \pm 0.006_{stat.} \pm 0.004_{syst.} \pm 0.013_{ext.}$$

in good agreement with ChPT prediction. The measurements of $a_0 - a_2$ that we have described in this chapter, are shown for comparison in Fig. 6.4.

6.2 Measurements of h

Actually, only two measurements of the quadratic slope parameter h exist: the first analysis of the $K_L^0 \rightarrow 3\pi^0$ quadratic coefficient was carried out in 1992 by the Experiment 731 (E731) at Fermilab (Ref. [44]) that collected $\sim 5 \cdot 10^6$ events and more recently by NA48 collaboration (Ref. [12]) in 2001 with a high statistic of $\sim 14.7 \cdot 10^6$ decays. These experiments used the parametrization of Eq. (1.1) that is a Taylor expansion about the center of the Dalitz plot (Ref. [3]): rescattering effects were not considered until Cabibbo in 2004 (Ref. [7]). In both experiments the slope parameter was measured by performing a fit on the R distribution (the distance from the center of the Dalitz plot) corrected for detector acceptance effect (Fig. 6.6). The results quoted by the two experiments and the PDG average are:

$$\begin{aligned} h_{E731} &= (-3.3 \pm 1.1_{stat.} \pm 0.5_{syst.}) \cdot 10^{-3} \\ h_{NA48} &= (-6.1 \pm 0.9_{stat.} \pm 0.5_{syst.}) \cdot 10^{-3} \\ h_{PDG} &= (-5.0 \pm 1.4) \cdot 10^{-3} \end{aligned}$$

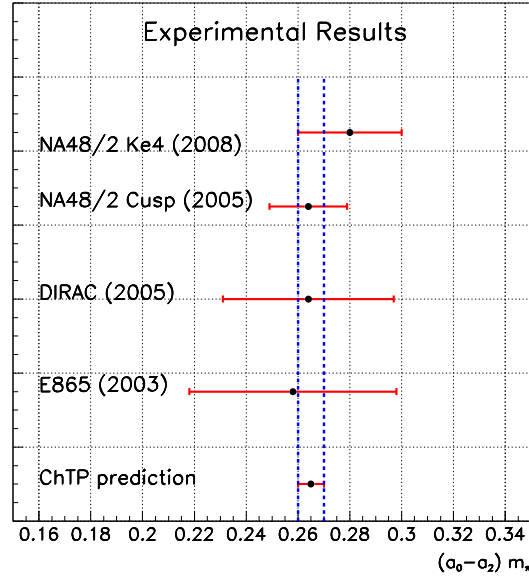


Figure 6.4: The $a_0 - a_2$ experimental results described in this chapter are superimposed in this plot.

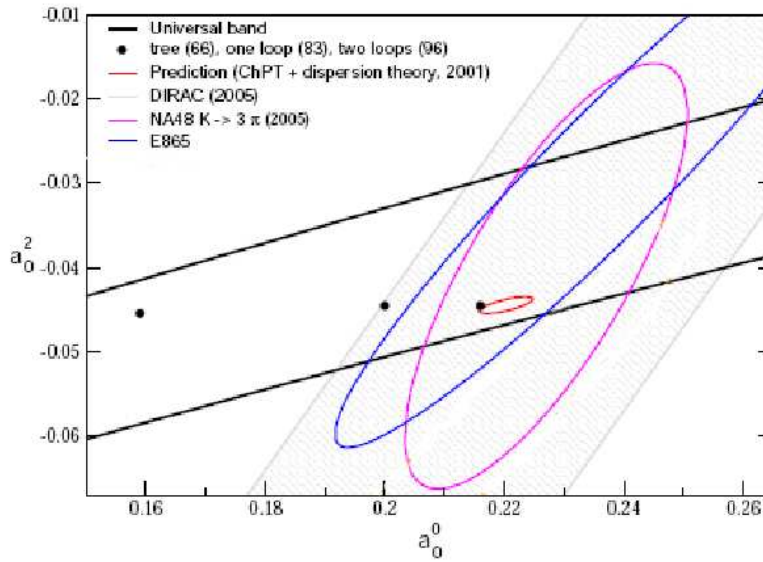


Figure 6.5: Comparison of the theoretical predictions of the scattering lengths and their measurements on the (a_0, a_2) plane (taken from Ref. [45]).

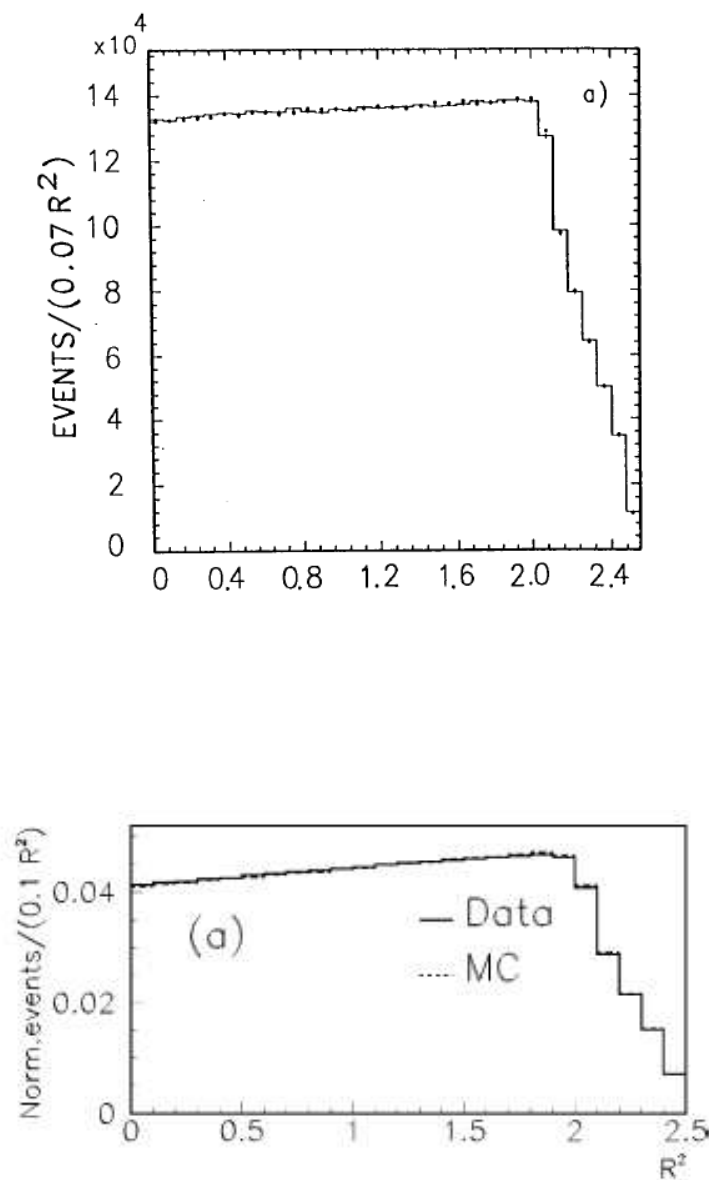


Figure 6.6: The R distribution (the distance from the center of the Dalitz plot) for E731 (upper figure) and NA48 (lower figure), taken from Ref. [44] and Ref. [12] respectively. In both figures the histogram is the data and the dashed line represents the simulated events. The fit of the slope parameter was performed in both data and MC distributions; in order to eliminate the effects of the acceptance, the value extracted by data had to be corrected so that: $h = h_{\text{data}} - h_{\text{MC}}$

Chapter 7

NA48 experiment

The NA48 is an experiment, installed at the CERN SPS accelerator, which took data from year 1997 until 2001 aiming at a substantial improvement in the accuracy on the double ratio:

$$R = \frac{BR(K_L \rightarrow \pi^0 \pi^0)}{BR(K_S \rightarrow \pi^0 \pi^0)} / \frac{BR(K_L \rightarrow \pi^+ \pi^-)}{BR(K_S \rightarrow \pi^+ \pi^-)}$$

a quantity directly correlated to CP symmetry violation.

NA48 makes use of a double K_L , K_S beam in order to have a simultaneous collection of the four categories of events entering in the double ratio (Fig. 7.1); the quasi collinearity of the K_L , K_S provided by the experiment, allows to minimize systematic differences.

In this chapter we give a general description of the K_L beam and the detectors used for the present analysis.

7.1 K_L^0 beam

The SPS delivers a 450 GeV protons beam in 2.5 pulses and a cycle time of 14.4 s. The primary high flux proton beam ($\sim 1.5 \times 10^{12}$ protons per pulse) impinges on the first K_L beam target, with an incident angle of 2.4 mrad relative to the K_L beam axis. The charged component of the outgoing particles produced is swept away by a bending magnet. The remaining neutral particles pass through a system of collimators. At the exit of the final collimator, placed at 126 m downstream the K_L target, the fiducial decay region begins and, at that point, only a small fraction of the most energetic long-lived component survives.

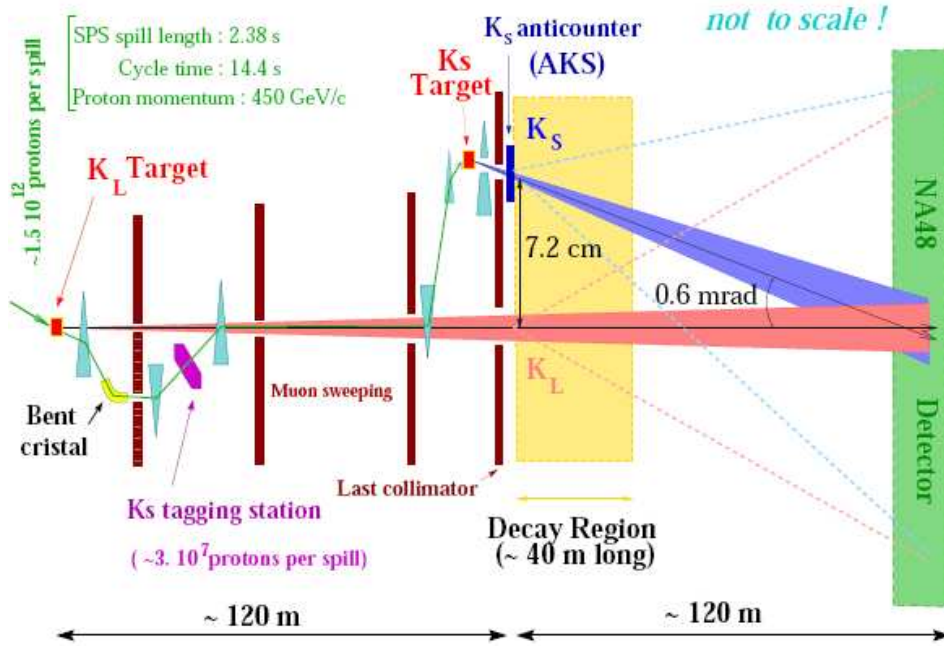


Figure 7.1: Sketch of the K_L and K_S beams: the two simultaneous and almost collinear kaon beamlines converge in the same center at the central detector longitudinal position.

7.2 Liquid Krypton electromagnetic calorimeter (LKr)

A liquid krypton calorimeter (LKr) is used to measure gamma energy, position and time and it is used to reconstruct $\pi^0 \rightarrow \gamma\gamma$ decays. It is an almost homogeneous ionization chamber with an active volume of $\sim 10\text{ m}^3$ of liquid krypton, segmented transversally into 13248 $2\text{ cm} \times 2\text{ cm}$ projective cells by a system of Cu-Be ribbon electrodes. The calorimeter is $27 X_0$ thick¹ and the energy resolution has been measured to be:

$$\frac{\sigma(E)}{E} = \frac{0.09}{E_{(\text{GeV})}} \oplus \frac{0.032}{\sqrt{E_{(\text{GeV})}}} \oplus 0.0042 \quad (7.1)$$

¹For krypton: $X_0 = 4.7\text{ cm}$

The space resolution for single electromagnetic shower can be parametrised as

$$\sigma_x = \sigma_y = \frac{0.42}{\sqrt{E_{(\text{GeV})}}} \oplus 0.06 \quad \text{cm} \quad (7.2)$$

for each transverse coordinate x, y .

A neutral hodoscope consisting of a plane of scintillating fibers is installed in the LKr calorimeter at a depth of $\sim 9.5 X_0$. It divides into four quadrants, each consisting of eight bundles of vertical fibers optically connected to photomultiplier tubes.

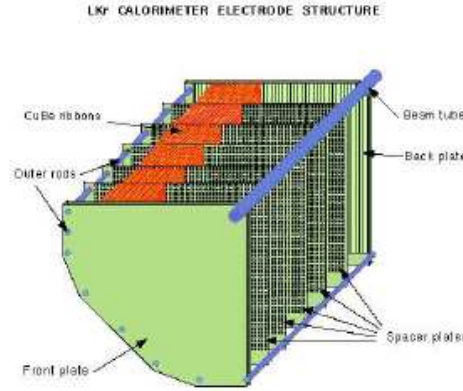


Figure 7.2: The electrode structure of the LKr calorimeter.

7.3 Neutral trigger

In order to handle large amount of data at high rate, a fast trigger response has to be taken in at most 200 ns , to decide if an event has to be recorded or not. The trigger for $3\pi^0$ decays operates on the sums of signals from groups of 2×8 cells of the calorimeter, in both horizontal and vertical orientation. The signals are digitized and summed into 64 columns and 64 rows providing the two projection of the deposited energy. The trigger system computes the energy sum ($m_0 = \sum_i^{64} E_i$), the first and second moments of the energy ($m_{1,x} = \sum_i^{64} x_i E_i$, $m_{2,x} = \sum_i^{64} x_i^2 E_i$ and the same for y projection) every 25 ns . Then, the moments are combined to provide decay vertex, center of gravity of the

energy distribution and proper decay time for the candidate kaons. A loose neutral trigger was obtained applying cuts on energy and lifetime of the kaon and on center of gravity. The efficiency of the neutral trigger was estimated with an independent trigger using the hodoscope installed in the LKr. The efficiency of the trigger has been measured for $K_L^0 \rightarrow 2\pi^0$ sample to be $\epsilon > 99.5\%$. In this work the same efficiency for $K_L^0 \rightarrow 3\pi^0$ has not been measured, we just check qualitatively the efficiency to be flat over the $K_L^0 \rightarrow 3\pi^0$ Dalitz plot.

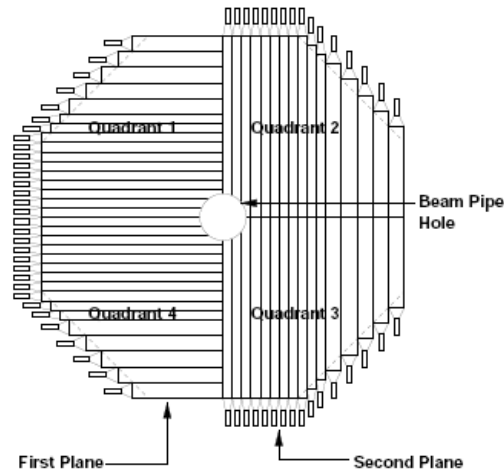


Figure 7.3: Layout of the neutral hodoscope.

Chapter 8

$K_L^0 \rightarrow 3\pi^0$ event reconstruction and acceptance correction

In this chapter we describe the selection criteria we have applied on the photons coming from $K_L^0 \rightarrow 3\pi^0 \rightarrow 6\gamma$ decays and how the geometrical and kinematical decay variables used in the present analysis have been reconstructed from the data. Finally, we list further requirements we have imposed on these quantities. After the reconstruction procedure, we have about $1.2 \cdot 10^8$ events passing all the requirements.

8.1 Selection criteria for the photons

The main selection on the data requires 6 LKr clusters satisfying the following conditions:

- the non linearity of the calorimeter at low energies is prevented, requiring the energy of each cluster has to be greater than 3 GeV; clusters having energies greater than 100 GeV are also rejected.
- in order contain the electromagnetic showers well inside the physical edges of the calorimeter, the x, y cluster coordinates satisfy: $|x|, |y| < R_{oct.} = 116 \text{ cm}$, $|x| + |y| < \sqrt{2} R_{oct.}$, $|x| < 63, 2 \text{ cm} \& |y| < 84, 7 \text{ cm}$, $|x| < 52, 2 \text{ cm} \& |y| < 94, 7 \text{ cm}$ and $(|x| - 63, 2)^2 + (|y| - 94, 7)^2 < 121 \text{ cm}$ (octagonal perimeter).
- the photons could be lost inside the beam-pipe ($\varnothing 8 \text{ cm}$) placed at the centre of the calorimeter, so that the distance from the cluster and the centre of setup has to be greater than 15 cm (Fig. 8.1)

- the distance between all cluster pairs has to be greater than 10 cm in order to avoid overlapping of the clusters caused by the electromagnetic showers
- to avoid energy losses in the LKr, the distance from the cluster to the closest dead calorimeter cell is required to be greater than 2 cm
- the radial position of the center-of-gravity

$$COG = \frac{\sqrt{(\sum_{i=1}^6 E_i x_i)^2 + (\sum_{i=1}^6 E_i y_i)^2}}{\sum_{i=1}^6 E_i}$$

where E_i , x_i and y_i are the energies and the positions of the six selected clusters, has to be less than 10 cm to reject background

The best photon pairing is done using a kinematical fit: for each combination of all 15 possible photon pairings and assuming the neutral kaon coming from the K_L^0 target and a $K \rightarrow 3\pi^0 \rightarrow 6\gamma$ decay, the cluster energies E_i are adjusted so that $E'_i \rightarrow E_i + \delta_i$ minimize:

$$\chi_{fit}^2 = \sum_{i=1}^6 \frac{(\delta_i)^2}{\sigma(E_i)^2}$$

where $\sigma(E_i)$ is the energy resolution. The fit finds improved values E'_i of the measured quantities which satisfy the $3\pi^0$ mass constraints. The degrees of freedom involved in the fit are: $d.o.f. = 6$ (energies for event) - 3 (mass constraints) = 3. The best pairing was obtained by taking the combination giving the minimum χ_{fit}^2 . The corresponding cut applied is $\chi_{fit}^2 < 15$. From now on the correct pairing is assumed to be (1, 2), (3, 4) and (5, 6).

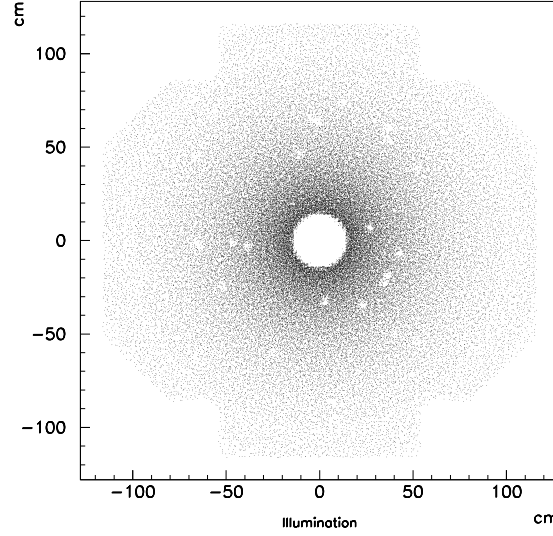


Figure 8.1: The photon illumination on the LKr calorimeter. The dead cells and the octagonal shape of the calorimeter are shown in the figure

8.2 Data reconstruction and further requirements

Using the coordinates (x_i, y_i) at the calorimeter and the energies (E_i) of the 6 photons, all the physical variables of interest for $K_L^0 \rightarrow 3\pi^0 \rightarrow 6\gamma$ decays are reconstructed.

- The decay vertex Z_{vertex} : is the longitudinal distance of the decay vertex from a reference point taken as the Z position of the K_S^0 target. First, the distance of the decay vertex D from the calorimeter is reconstructed by using the kinematical constraint

$$M_{K_L^0} = \sqrt{\left(\sum_{i=1}^6 E_i\right)^2 - \left(\sum_{i=1}^6 \vec{P}_i\right)^2} = \sqrt{\sum_{i=1}^6 \sum_{j>i}^6 2 E_i E_j (1 - \cos(\theta_{ij}))}$$

and making the small angles approximation $1 - \cos(\theta_{ij}) \sim \theta_{ij}^2/2 = (r_{ij}/D)^2/2$:

$$D = \frac{\sqrt{\sum_i^6 \sum_{j>i}^6 E_i E_j r_{ij}^2}}{M_{K_L^0}}$$

where r_{ij} is the distance between the i, j clusters on the LKr plane. Finally,

$$Z_{vertex} = Z_{LKr} - \frac{\sqrt{\sum_{i=1}^6 \sum_{i>j}^6 E_i E_j [(x_i - x_j)^2 + (y_i - y_j)^2]}}{M_{K_L^0}}$$

where Z_{LKr} is the longitudinal position of the LKr calorimeter. No cuts are applied on the reconstructed decay vertex.

- The invariant masses $M_{\pi^0\pi^0}^{(1)}, M_{\pi^0\pi^0}^{(2)}, M_{\pi^0\pi^0}^{(3)}$: for each event, the three $\pi^0\pi^0$ invariant masses are reconstructed by the same method used for the Z_{vertex} . For example, if a $\pi^0\pi^0$ couple is labelled by the pairing (1, 2) and (3, 4), the decay distance to the LKr is calculated as the average of the decay vertices of the two π^0 :

$$D = \frac{1}{2m_{\pi^0}} (\sqrt{E_1 E_2 r_{12}^2} + \sqrt{E_3 E_4 r_{34}^2})$$

The invariant mass of the couple is then:

$$M_{\pi^0\pi^0}^{(1)} = \frac{\sqrt{\sum_{i=1}^4 \sum_{i>j}^4 E_i E_j [(x_i - x_j)^2 + (y_i - y_j)^2]}}{D}$$

In Fig.8.2, $M_{\pi^0\pi^0}^2(Gen.)$ versus $M_{\pi^0\pi^0}^2(Gen.) - M_{\pi^0\pi^0}^2(Rec.)$ is shown. In order to estimate the $M_{\pi^0\pi^0}^2$ resolution, we created projections onto the $M_{\pi^0\pi^0}^2(Gen.) - M_{\pi^0\pi^0}^2(Rec.)$ axis, in $M_{\pi^0\pi^0}^2(Gen.)$ -slices; the resolution for a given slice is the r.m.s. obtained by fitting the projected distribution with a Gaussian. The resolution is the best at small $M_{\pi^0\pi^0}^2$ in particular, near the cusp point the resolution is $\sigma_{4m_{\pi^\pm}^2} = 1.3 \cdot 10^{-4} GeV^2$.

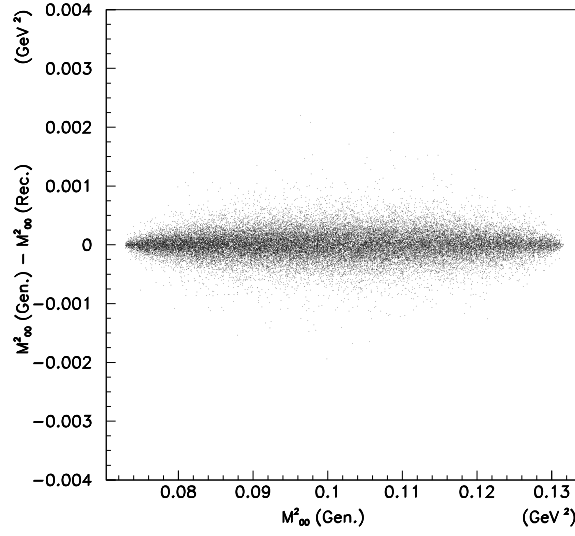


Figure 8.2: *The histogram shows $M^2_{\pi^0\pi^0}(Gen.)$ versus $M^2_{\pi^0\pi^0}(Gen.) - M^2_{\pi^0\pi^0}(Rec.)$ (sorted at random)*

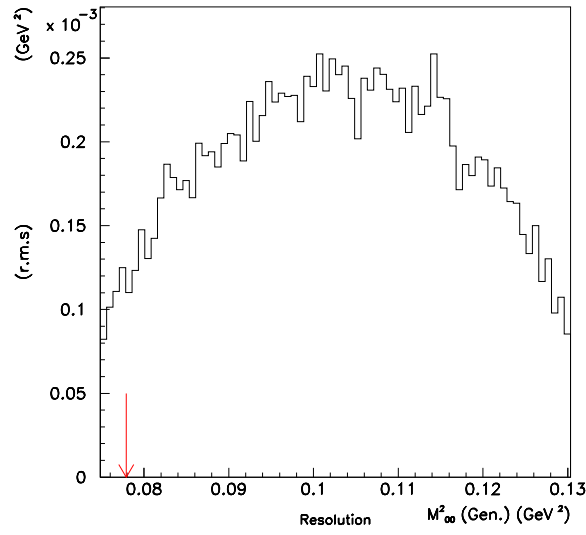


Figure 8.3: *$M^2_{\pi^0\pi^0}$ resolution versus generated $M^2_{\pi^0\pi^0}$. This plot represent the width of the sigar-shaped previous plot as function of $M^2_{\pi^0\pi^0}$. The cusp point is indicated by the arrow.*

- Kaon energy E_K : The kaon energy is the sum of the energies of 6 photons passing all the requirements:

$$E_K = \sum_{i=1}^6 E_{\gamma}^i$$

The cut applied on the kaon energy is: $70 \text{ GeV} < E_K < 170 \text{ GeV}$

- The decay proper time τ : the proper time τ , measured in unit of K_S^0 mean life ($c\tau_{K_S^0} = 2.676 \text{ cm}$), is reconstructed using the decay vertex and the momentum of the kaon:

$$\frac{\tau}{\tau_{K_S^0}} = \frac{(Z_{vertex} - Z_{AKS})}{\tau_{K_S^0} \gamma \beta} = \frac{(Z_{vertex} - Z_{AKS}) \cdot M_{K_L^0}}{\tau_{K_S^0} P_K}$$

where, by convention, the decay vertex is referred to the AKS position (Z_{AKS} is the longitudinal distance of the AKS to the K_S^0 target). The requirement on the proper time is: $\tau/\tau_{K_S^0} > 0.8 + 0.06 \cdot E_K$. This cut makes the acceptance more flat over the Dalitz plot.

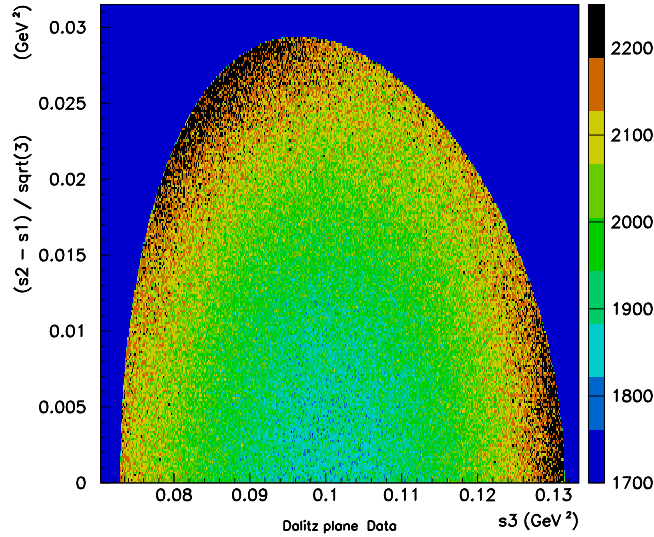


Figure 8.4: The data on the Dalitz plane ($s_3, \frac{|s_1 - s_2|}{\sqrt{3}}$). The binning of axes is chosen so that a pixel has size $1\sigma \times 1\sigma$ (the plot is deformed due to the different scale of two axes)

8.3 Background rejection

8.3.1 Combinatorial background

Due to energy resolution the pairing of photons into neutral pions may be wrong in a certain fraction of cases which have been estimated by Monte Carlo to be at per mille level. A good event contains 6 electromagnetic showers in the calorimeter so that there are 15 different ways of pairing them. For each pairing our kinematical fit returns a χ^2 value. The best pairing is defined to be the one associated to the lowest χ^2 value. We have defined a new variable $\Delta\chi^2$ as the difference between the second best and the best χ^2 . As shown in Fig. 8.5, $\Delta\chi^2$ exhibits an excess of events at low values. These events are rejected because they are ambiguous from the pairing point of view. The corresponding cut applied is $\Delta\chi^2 < 20$. This cut rejects a small fraction of events which is a small fraction with respect to our full statistics of $1.2 \cdot 10^8$.

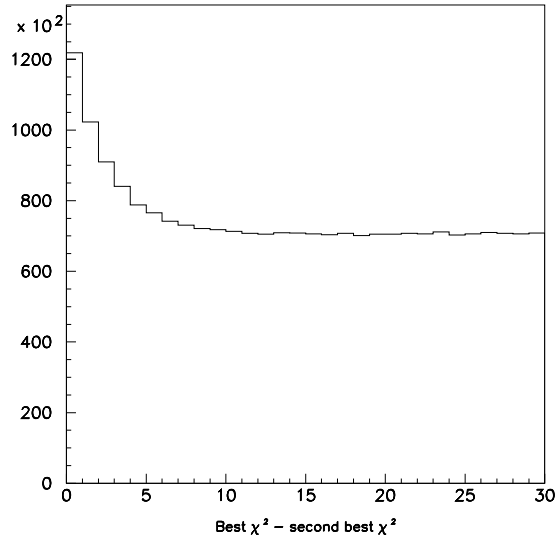


Figure 8.5: $\Delta\chi^2$ distribution. The ambiguous pairings are at low values. In order to reject these pairings we have applied the cut $\Delta\chi^2 < 20$

8.3.2 π^0 Dalitz decay

The π^0 is known to decay into a single photon plus a couple of electrons $\pi^0 \rightarrow e^+e^-\gamma$ (Dalitz decay) with a branching ratio $B.R. = 1.2\%$ (Ref. [13]). Therefore, our $3\pi^0$ final state may produce electron with a probability of 3% which is not totally negligible. It would be easy to reject them with a charge track reconstruction but unfortunately in year 2000 the NA48 spectrometer was removed from the experiment because it was in repair¹. A new technique to reject the pairs by exploiting the non zero value of the spectrometer magnetic field has been applied. First, we recall that the invariant dilepton mass distribution $m_{e^+e^-}$ for the Dalitz decay is given by the Kroll-Wada formula (Ref. [46]):

$$\frac{d\Gamma}{dx} = \frac{\alpha}{3\pi} \Gamma_{\pi^0\gamma\gamma} \frac{(1-x)^3}{x} \sqrt{1 - \frac{r}{x}} \left(2 + \frac{r}{x}\right)$$

where $x = \frac{m_{e^+e^-}^2}{m_\pi^2}$ and $r = \frac{4m_e^2}{m_\pi^2}$. According to this formula, the Dalitz pair is emitted nearly collinear with a very tiny opening angle. The pairs are deviated trasversally by the spectrometer magnetic field that provide for a nominal kick momentum of about² $P_{\text{kick}} = 0.240 \text{ MeV}$ (Fig. 8.6). The distance between the impact points of the two electrons on the LKr plane is $\Delta y \sim 0$ and $\Delta x = P_{\text{kick}} D \left(\frac{1}{E_{e^-}} + \frac{1}{E_{e^+}} \right)$.

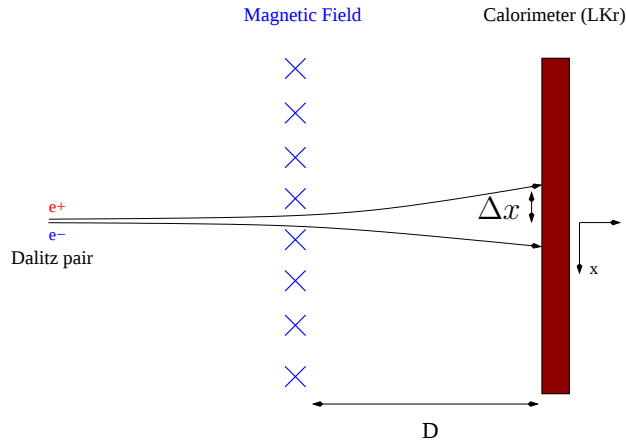


Figure 8.6: *Dalitz decay geometry. The e^+e^- couple is emitted nearly collinear so that a Dalitz decay event is identified on the LKr plane by a small relative longitudinal distance $\Delta y \sim 0$ and a trasversal distance $\Delta x = P_{\text{kick}} D \left(\frac{1}{E_{e^-}} + \frac{1}{E_{e^+}} \right)$.*

¹During winter 99-2000 the Kevlar window insulating the drift chamber system from outside broke.

²Although the spectrometer was in repair, the corresponding magnetic field was on.

The dilepton system acquires a “fake” invariant mass due to P_{kick} (see sec. 8.2):

$$m_{e^+e^-}^{2, \text{LKr}} = \frac{E_{e^-} E_{e^+}}{D^2} \cdot r_{e^+e^-}^2$$

and the e^+e^- distance on the LKr is:

$$r_{e^+e^-}^2 \simeq (\Delta x)^2 = \left[\left(x + \frac{P_{\text{kick}}}{E_{e^+}} D \right) - \left(x - \frac{P_{\text{kick}}}{E_{e^-}} D \right) \right]^2 = P_{\text{kick}}^2 D^2 \left(\frac{1}{E_{e^-}} + \frac{1}{E_{e^+}} \right)^2$$

so that

$$P_{\text{kick}} = \frac{m_{e^+e^-}^{\text{LKr}}}{\sqrt{2 + \frac{E_{e^+}}{E_{e^-}} + \frac{E_{e^-}}{E_{e^+}}}} \quad (8.1)$$

In Fig. 8.8 it is shown the P_{kick} distribution before the event selection, for those events having at least two clusters at $\Delta y < 4\text{ cm}$. This distribution exhibits an excess of events around $P_{\text{kick}} = 0.268\text{ MeV}$ corresponding mainly to Dalitz pairs.

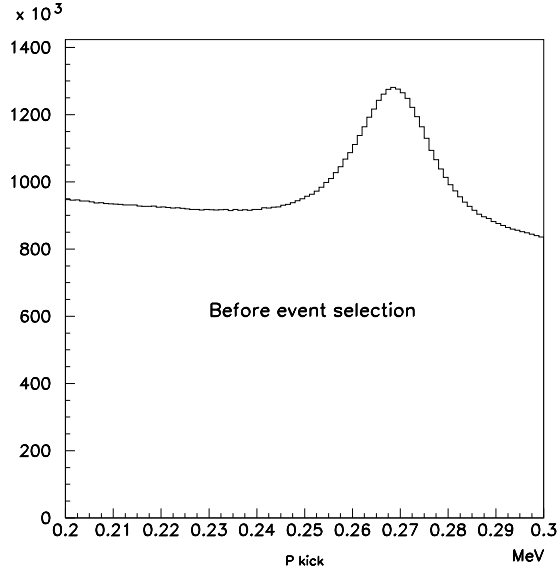


Figure 8.7: *The P_{kick} distribution before applying the event selection. The excess of events near $P_{\text{kick}} = 0.268\text{ MeV}$ is due to the Dalitz decays.*

However, the background due to the Dalitz decays is strongly suppressed by the selection criteria we have applied to the data, as shown in Fig. 8.8:

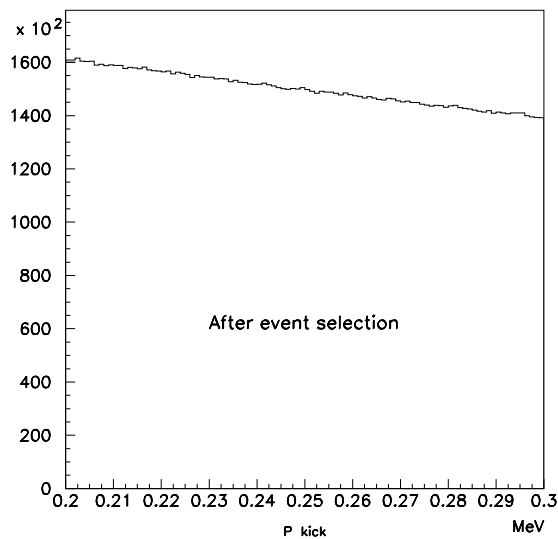


Figure 8.8: *The P_{kick} distribution after the event selection. The Dalitz decays are rejected by applying the selection criteria.*

8.4 MC to data comparison

A geometrical Monte Carlo is used to reproduce the K_L^0 decay in NA48 experiment. In order to estimate the detector acceptance about $4.5 \cdot 10^8$ MC events are generated that pass all selection criteria required for the data with no rescattering and $h = 0$ i.e., pure phase space. The kaon momentum is generated by using an empirical probability distribution for the momentum spectrum:

$$S(P_K) = N \cdot \exp(-a \cdot P_K) \cdot P_K^b$$

where $a = 1/30.45$ and $b = 1.671$ are the slopes parameters extracted from the experimental spectrum and N is the normalization factor. The longitudinal position of the decay vertex Z_{vertex} is created in a similar way by using the exponential decay law and the generated P_K :

$$P(Z) = N \cdot \exp(-Z/Z_0)$$

where $Z_0 = \tau_{K_L^0} \cdot P_K/M_{K_L^0}$ and $c\tau_{K_L^0} = 1554$ cm. The $K_L^0 \rightarrow 3\pi^0 \rightarrow 6\gamma$ decays are simulated by choosing randomly the angles that define the decay tracks and the $3\pi^0$ energies, according to the $M_{\pi^0\pi^0}$ distribution given by the phase space and kinematical constraints. Then, the generated photons are propagated to calorimeter. The effect of calorimeter resolution is simulated by smearing the photon energies and positions for the detector resolutions $\sigma(E)$ and σ_x . In the simulation a parametrisation of the photon energy non-Gaussian tails is contained. So far, Monte Carlo does not simulate the π^0 Dalitz decay $\pi^0 \rightarrow e^+e^-\gamma$. However, we rejected this background from data and we applied the same cuts also in the simulation. The histograms of the most important kinematical variables, used to check the quality of the Monte Carlo simulation, are shown in Fig. 8.13-8.12 (all plots are normalized to the same area).

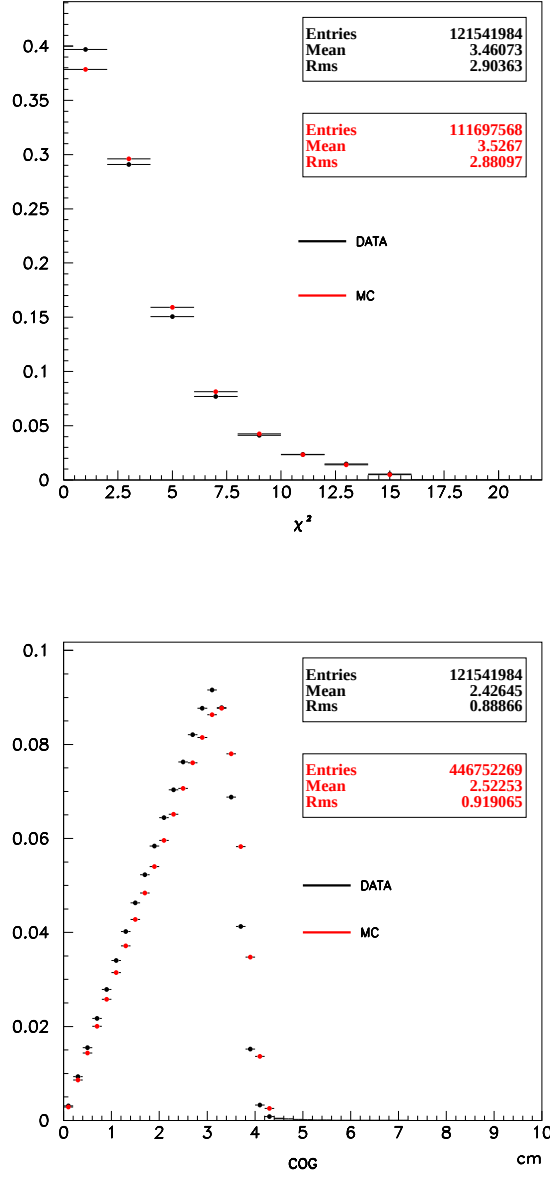


Figure 8.9: *Data - MC comparison: (Upper plot) χ_{fit}^2 given by the kinematical fit. (Lower plot) center-of-gravity (COG): the discrepancy between data and MC is due to the fact that the beam divergence is to be fine-tuned in MC.*

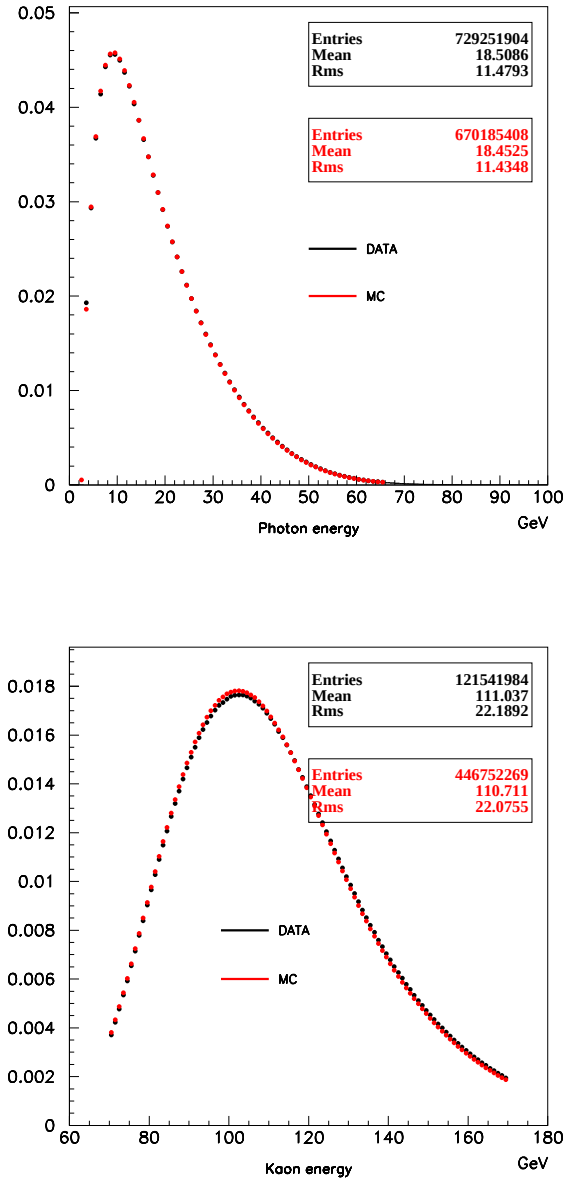


Figure 8.10: *Data - MC comparison for the energies: (Upper plot) the energies of the 6 gammas are plotted for each event; (Lower plot) K_L^0 energy distribution. The agreement between data and MC is good in both histograms*

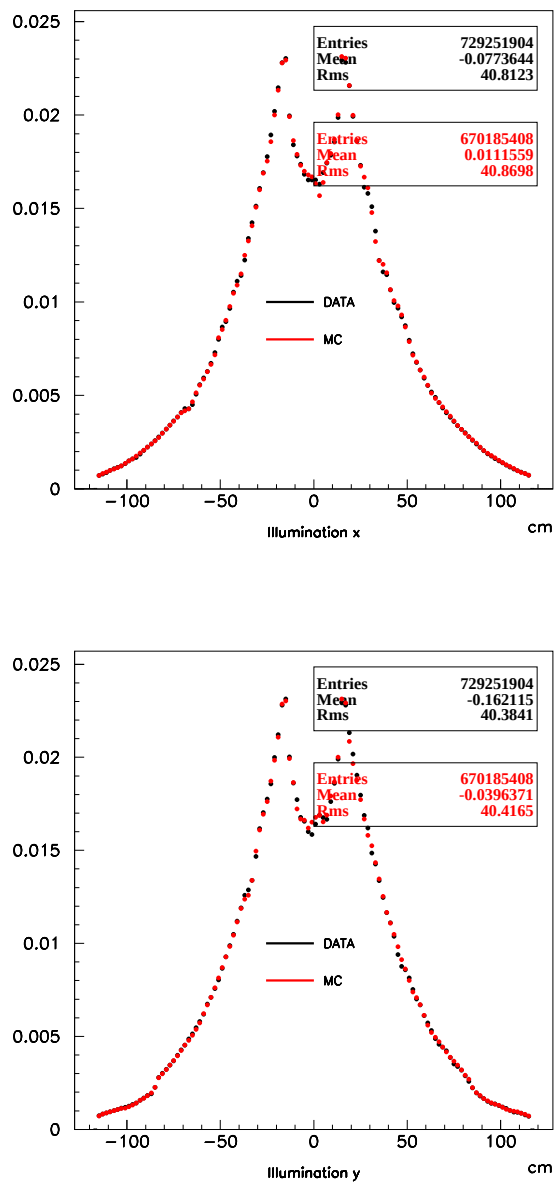


Figure 8.11: *Data - MC comparison for the illumination on the calorimeter plane: (Upper plot) illumination projection on the x-axis; (Lower plot) illumination projection on the y-axis*

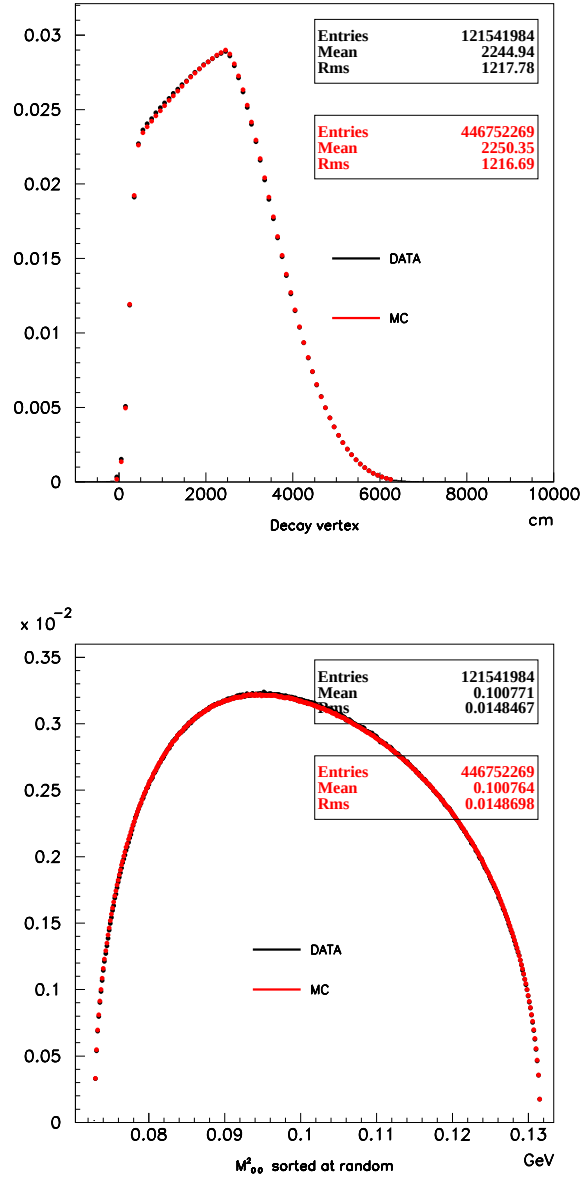


Figure 8.12: *Data - MC comparison: (Upper plot) distance of the decay vertex Z_{vertex} to LKr; (Lower plot) invariant mass $M_{\pi^0\pi^0}^2$ sorted at random*

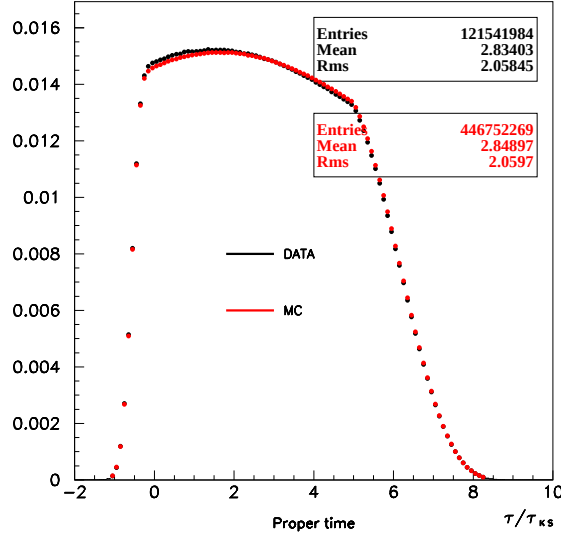


Figure 8.13: *Data - MC comparison for the kaon proper time τ/τ_{KS} . The negative values are due to the choice of AKS position as reference point for the decay vertex*

8.5 Detector acceptance

Data have to be corrected for detector acceptance. The acceptance, plotted on the $(s_3, \frac{|s_1-s_2|}{\sqrt{3}})$ plane, was estimated as follows (Fig. 8.14) :

$$\text{Acceptance} = \frac{\text{Dalitz plot REC}}{\text{Dalitz plot GEN}}$$

The detector efficiency for the $K_L^0 \rightarrow 3\pi^0$ decay increases from the centre to the external perimeter of the Dalitz plot, varying between 2.8% (at the centre) and 4.0% (at the edge). Moreover the acceptance has, with good accuracy, a radial symmetry (i.e. is independent from the angle $\theta = \arctan(\frac{|s_1-s_2|}{\sqrt{3}\cdot s_3})$). The acceptance and the resolution modify the physical event distribution on the Dalitz plane (fig.8.2). Their effects have to be corrected for in order to extract physical informations from experimental data. We define the acceptance-resolution matrix (pixel-spread-function) as:

$$A_{(ij),(kl)} = \frac{N_{ij \rightarrow kl}^{\text{MC}}}{N_{ij}^{\text{MC}}} \quad (8.2)$$

where N_{ij}^{MC} are the MC events generated at pixel (i, j) and $N_{ij \rightarrow kl}^{\text{MC}}$ the events generated at (i, j) and reconstructed at (k, l) .

$A_{(ij),(kl)}$ represents the probability for an event generated at pixel (i, j) to be reconstructed at pixel (k, l) . The error $\delta A_{(ij),(kl)}$ is evaluated according to binomial variance:

$$\delta A_{(ij),(kl)} = \sqrt{\frac{A_{(ij),(kl)}(1 - A_{(ij),(kl)})}{N_{ij}^{\text{MC}}}} \quad (8.3)$$

In order to handle with the pixel-spread-function, we fitted the Dalitz plot by using a reduced binning with respect to Fig. 8.4. Then we chose to fit pixels with size $5\sigma \times 5\sigma$.

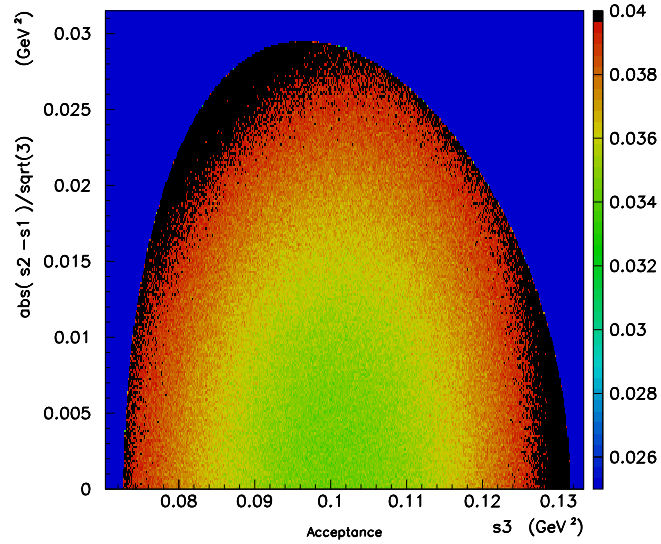


Figure 8.14: *Detector acceptance using a geometrical MC: with a good accuracy, the acceptance has a radial symmetry.*

Chapter 9

Fitting methods and parameters evaluation

9.1 Fit of the matrix element $|\mathcal{M}_{000}(x, y)|^2$

The density function on the Dalitz plane for the $K_L^0 \rightarrow 3\pi^0$ decays is the matrix element squared:

$$\rho(x, y) \propto |\mathcal{M}_{000}(x, y, a_0 - a_2, a_2, h)|^2$$

where, as usual, $x = s_3$, $y = |s_1 - s_2|/\sqrt{3}$ are the two independent variables of the Dalitz plot (Fig. 8.4). The theoretical models used for $\mathcal{M}_{000}$ are reported in part I, Eq. (3.16)-(3.17). The fit has 4 free parameters: $a_0 - a_2$, a_2 , h and an irrelevant overall normalization λ . In the limit of exact isospin symmetry, all scattering lengths combinations can be expressed as linear functions of $a_0 - a_2$ and a_2 (Eq. (3.8-3.9)). However, we know that the cusp effect is quite insensitive to a_2 so that it was fixed at the value of Ref. [10].

In order to perform the 2-dimensional fit, we have to correct the theoretical rate for the detector acceptance and resolution (pixel migration) both coded in the pixel-spread-function $A_{(i,j)(k,l)}$, the probability for an event generated at pixel (i, j) to be reconstructed at (k, l) . The convolution of $\rho(x, y)$ with the pixel-spread-function gives the Dalitz plot density corrected for the detector acceptance and pixel migration:

$$\rho_{ij}^{\text{pred.}} \equiv \rho^{\text{pred.}}(x_i, y_j) = \sum_{k,l} \rho(x_k, y_l) \cdot A_{(kl),(ij)}$$

where x_i, y_j are evaluated at the center of the pixel (i, j) .

The χ^2 to minimize is then:

$$\chi^2 = \sum_{i,j} \frac{(N_{ij}^{\text{data}} - \lambda \cdot \rho_{ij}^{\text{pred.}})^2}{(\sigma_{ij}^{\text{data}})^2 + (\lambda \sigma_{ij}^{\text{pred.}})^2} \quad (9.1)$$

where the sum is extended to a determined fit region and λ is an irrelevant number accounting for the overall normalization. The uncertainties $\sigma_{ij}^{\text{data}}$ and $\sigma_{ij}^{\text{pred.}}$ are estimated as follows:

$$(\sigma_{ij}^{\text{data}})^2 = N_{ij}^{\text{data}} \quad (9.2)$$

$$(\sigma_{ij}^{\text{pred.}})^2 = \sum_{k,l} \rho_{kl}^2 \cdot \delta A_{(kl),(ij)}^2 \quad (9.3)$$

where $\delta A_{(kl),(ij)}$ is the error on the pixel-spread-function given by Eq. (8.3).

9.1.1 External parameters

We report the external parameters and their values used in the CI model and employed in the fit (table 9.1):

Parameters		Value
Effective ranges:	r_0	1.25 ± 0.04 [31]
	r_2	1.81 ± 0.05 [31]
$K_L^0 \rightarrow \pi^+ \pi^- \pi^0$ slopes:	linear: g_L^+	0.678 ± 0.008 [13]
	quadratic: h_L^+	0.076 ± 0.006 [13]
Isospin amplitudes ratio:	A_L^+/A_L^0	0.342 ± 0.002

Table 9.1: *The values of the external parameters involved in the CI model. The value obtained for A_L^+/A_L^0 is discussed below*

In the limit of exact isospin symmetry ($m_{\pi^0} = m_{\pi^+}$) and assuming $\Delta I = 1/2$ rule, the predicted value for A_L^+/A_L^0 should be $1/3$ (Appendix A); the experimental value of the isospin amplitudes ratio is obtained by taking into account the differences in phase-space volumes due to the mass splitting and the differences in slopes. We have computed the ratio:

$$\frac{\Gamma^{PDG}(K_L^0 \rightarrow \pi^+ \pi^- \pi^0)}{\Gamma^{PDG}(K_L^0 \rightarrow 3\pi^0)} = 3! \cdot \frac{|A_L^+|^2 \cdot \int |\mathcal{M}_{+-0}(s_1, s_2, s_3)|^2 d\phi_+^{(3)}}{|A_L^0|^2 \cdot \int |\mathcal{M}_{000}(s_1, s_2, s_3)|^2 d\phi_0^{(3)}}$$

where we have used the PDG slope values and the PDG parametrisation for $\mathcal{M}_{+-0}$ and $\mathcal{M}_{000}$, neglecting the small rescattering terms; moreover we have explicitated the $3!$ coming from the $3\pi^0$ permutations. $\mathcal{M}_{+-0}$ and $\mathcal{M}_{000}$ are normalised such that at the center of the Dalitz plot $\mathcal{M}_{+-0}^{\text{center}} = \mathcal{M}_{000}^{\text{center}} = 1$. From

$$\frac{\Gamma^{PDG}(K_L^0 \rightarrow \pi^+\pi^-\pi^0)}{\Gamma^{PDG}(K_L^0 \rightarrow 3\pi^0)} = \frac{(12.56 \pm 0.05)\%}{(19.56 \pm 0.14)\%} = 0.642 \pm 0.005$$

we have obtained:

$$\frac{A_L^+}{A_L^0} = \sqrt{\frac{0.642^{BR} \times 1.100^{ph.sp.}}{3!}} = 0.342 \pm 0.002$$

These parameters are sources of the external error that will be estimated in the following.

9.2 Fit using the leading order: a first result of the CI model

The matrix element at the leading order is given by Eq. (3.16) and it requires 3 free parameters: the scattering lengths combination $a_0 - a_2$, the quadratic slope h and the normalisation factor λ (Fig. 9.1). The fit results using the PDG parametrisation and the leading order are shown in table 9.2- 9.3, respectively:

PDG parametrisation	
h ($\times 10^{-3}$)	$\chi^2/d.o.f$
$-7.5 \pm 0.2_{(stat)}$	$2243.5/2037 \sim 1.10$

Table 9.2: *The value for h using the PDG parametrisation*

Leading order			
$a_0 - a_2$ (m_{π^+} unit)	h ($\times 10^{-3}$)	$\chi^2/d.o.f$	Correlation $a_0 - a_2/h$
$0.227 \pm 0.017_{(stat)}$	$-6.3 \pm 0.2_{(stat)}$	$2068.2/2036 \sim 1.02$	40%

Table 9.3: *The values for $a_0 - a_2$ and h using the leading order*

We have excluded 5 bins below the Dalitz plot boundary because we have evidence that the Monte Carlo simulation of these bins is critical. The bad value of the χ^2 obtained arises mainly from the region $M_{\pi^0\pi^0} < 2m_{\pi^\pm}$ as shown in Fig. 9.2. In view of the recent better understanding of the final state rescattering process, the PDG parametrisation has become obsolete. When the naive PDG parametrisation is replaced with a more sophisticated formula taking into account final state rescattering at leading order, the χ^2 turns to be quite better. At leading order the scattering lengths combination $a_0 - a_2$ is below the expected value and is not consistent with the NA48/2 measurement on K^+ (Ref. [10]). Moreover, the value of h is definitely non zero and in agreement with previous experiments (Chapter 6).

We have presented this analysis at the leading order to demonstrate that the same Cabibbo-Isidori rescattering model, implemented with success in the $K^+ \rightarrow \pi^+\pi^0\pi^0$ decays, can explain the sudden change of slope in the $\pi^0\pi^0$ invariant mass distribution near $M_{\pi^0\pi^0}^2 = 4m_{\pi^\pm}^2$ visible in the K_L^0 system.

A better data-fit agreement could in principle be achieved by using the next-to-leading order approximation. The fit using this more complete model will be discussed in the following sections.

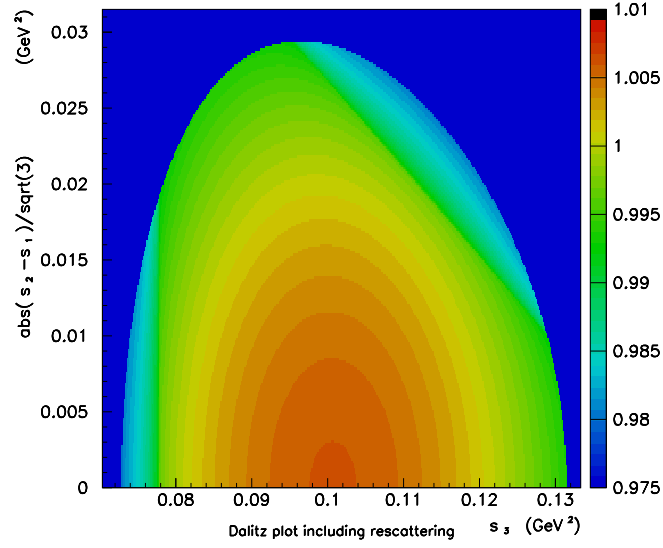


Figure 9.1: *The Dalitz plot at the leading order drawn with the values of scattering lengths and slope taken from Ref. [10, 13]. The sudden change in color is due to the cusp effect.*

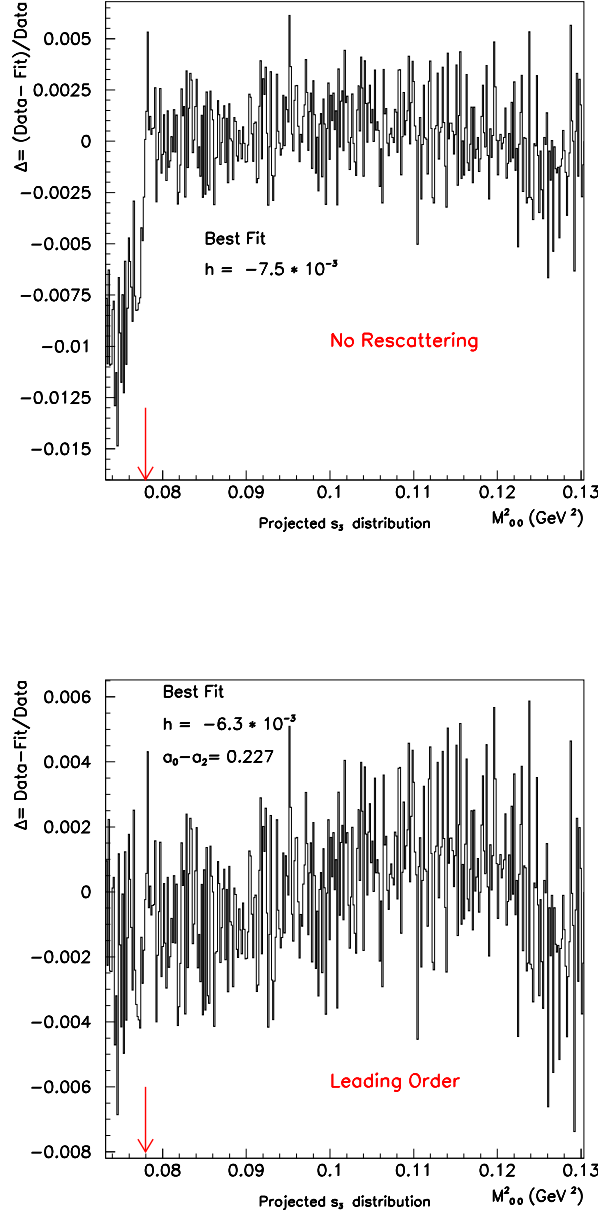


Figure 9.2: Upper figure: $\Delta = (\text{Data} - \text{Fit})/\text{Data}$ versus $M^2_{\pi^0\pi^0}$ using the PDG parametrisation: the cusp effect is clearly visible at the expected point (the cusp point $M^2_{\pi^0\pi^0} = 4m_{\pi^\pm}^2$ is indicated by the red arrow). Lower figure: $\Delta = (\text{Data} - \text{Fit})/\text{Data}$ versus $M^2_{\pi^0\pi^0}$ at the leading approximation: the sudden change in slope below the cusp is damped by rescattering terms

9.3 Fit using the next-to-leading order

The next-to-leading approximation for the matrix element taking into account the 2-loops contributions is reported in Eq. (3.17). Table 9.4 shows the fit result when $a_0 - a_2$ and h are free parameters (a_2 is fixed at the value of Ref. [31]):

Next-to-leading order			
$a_0 - a_2$ (m_{π^+} unit)	h ($\times 10^{-3}$)	$\chi^2/d.o.f$	Correlation $a_0 - a_2/h$
$0.180 \pm 0.012_{(stat)}$	$-3.5 \pm 0.5_{(stat)}$	$2078.5/2036 \sim 1.02$	90%

Table 9.4: *The values for $a_0 - a_2$ and h using the next-to-leading. At this order, the parameters $a_0 - a_2$ and h are strongly correlated*

This strong correlation found is quite unacceptable and suggests a redundancy in the number of free parameters. The slope h is a phenomenological parameter inspired by the old PDG parametrisation of the amplitude and it has no clear physical meaning¹. In fact, we are confident that the old non zero values for h , found by previous measurements, were in reality an artefact. Previous h values are indeed a reabsorbition of the rescattering process. If we fix the scattering lengths $a_0 - a_2$ and a_2 at the values of Ref. [10]) and we take the rescattering model for granted, we can do the exercise to fit h just in order to check if h parameter has an intrinsic meaning beyond the rescattering process. The result of this fit is shown in table 9.5:

Next-to-leading order ($a_0 - a_2$ fixed)	
h ($\times 10^{-3}$)	$\chi^2/d.o.f$
$+0.23 \pm 0.22_{(stat)}$	$2133.7/2037 \sim 1.05$

Table 9.5: *The value for h fixing $a_0 - a_2$ on the whole Dalitz plot. It should be noted that the value of h is compatible with zero*

¹In the past there were several attempts to estimate h . These predictions were obtained by Sawyer and Wali (Ref. [6] and Appendix A), Kambor *et al.* (Ref. [47]) and D'Ambrosio *et al.* (Ref. [48]) neglecting rescattering effects and their results range from $h = 0 \div 10^{-3}$. From the present analysis room from h seems to be left only at 10^{-4} level.

This result is clearly compatible with zero. This confirms that the parameter h is unnecessary in the parametrisation of the amplitude and it will be set to zero in the following.

If the slope parameter h is fixed at zero, as the previous result suggests, the scattering lengths combination is the only physical parameter in the fit. The value that we found for $a_0 - a_2$ is shown in table 9.6:

Next-to-leading order ($h = 0$ fixed)	
$a_0 - a_2$ (m_{π^+} unit)	$\chi^2/d.o.f$
$0.249 \pm 0.004_{(stat)}$	$2120.6/2037 \sim 1.04$

Table 9.6: *The value for $a_0 - a_2$ fixing $h = 0$ on the whole Dalitz plot.*

In Fig. 9.3 it is shown the $M_{\pi^0\pi^0}^2$ distribution to MC phase space ratio using the PDG prediction and the best fit result, respectively. In the next section we estimate the external error and we quote the final results obtained for $a_0 - a_2$.

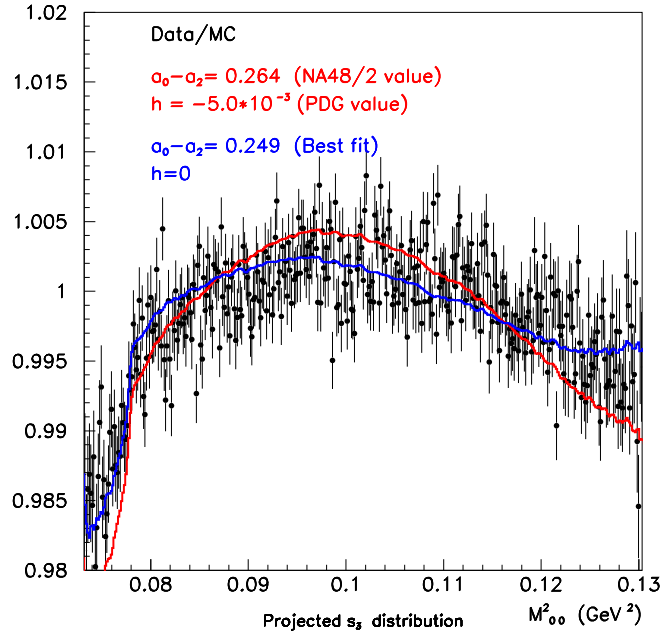


Figure 9.3: *Projected Dalitz distribution (on s_3 axis)/MC (phase space): the black cross are data; the red curve is the 2-loop prediction with the PDG value $h = -5.0 \cdot 10^{-3}$ and the NA48/2 value $a_0 - a_2$; the blue curve is drawn with the best fit value $a_0 - a_2 = 0.249$ and $h = 0$.*

9.4 External uncertainties and final results

The central value for $a_0 - a_2$ that we quote is given in the previous section when h is set at zero throughout the Dalitz plot. The external errors on $a_0 - a_2$, obtained by varying the parameters according to table 9.1, are listed in table 9.7:

External parameters	uncertainty on $a_0 - a_2$ (m_{π^+} unit)
a_2 (-0.044)	± 0.001
g_L^+ (0.678)	± 0.002
h_L^+ (0.076)	± 0.001
r_0 (1.25), r_2 (1.81)	$\pm 0.002, < 0.001$
A_L^+/A_L^0 (0.342)	± 0.001
(sub-total)	± 0.004

Table 9.7: *External errors on $a_0 - a_2$: the external uncertainty was estimated by varying one external parameter at time. The sub-total means that the listed errors are summed in quadrature*

The final result for the physical parameter $a_0 - a_2$ extracted from the $K_L^0 \rightarrow 3\pi^0$ Dalitz plot is:

$$m_{\pi^+}(a_0 - a_2) = 0.249 \pm 0.004_{(stat)} \pm 0.004_{(ext)} \pm 0.013_{(th.)} \quad (9.4)$$

$$\chi^2/n.d.f = 2120.6/2037 \sim 1.04$$

where we included the 5% theoretical error estimated by Cabibbo-Isidori (Ref. [8]). The result is consistent with the NA48/2 value at the level of 1σ .

Chapter 10

Summary and conclusions

We studied the $K_L^0 \rightarrow 3\pi^0$ Dalitz plot density described by the Cabibbo-Isidori model accounting for $\pi\pi$ rescattering effects in the final states. This model allows to extract the scattering lengths combination $a_0 - a_2$ and the slope parameter h .

We reconstructed about $1.2 \cdot 10^8$ $K_L^0 \rightarrow 3\pi^0$ events collected by NA48 experiment, taking into account the background due to photons pairing and Dalitz decays $\pi^0 \rightarrow e^+e^-\gamma$. Moreover we used a geometrical Monte Carlo to estimate the detector acceptance and resolution.

The fit was first performed fitting both h and $a_0 - a_2$ on the whole Dalitz plot using the next-to-leading order; the strong correlation between parameters (about 90%) suggested that the slope h was unnecessary for parametrising the amplitude. In fact by fixing $a_0 - a_2$ at the NA48/2 value we found:

$$h = (+0.23 \pm 0.20) \cdot 10^{-3}$$

that is clearly consistent with zero. The result for $a_0 - a_2$ without the parameter h was then:

$$m_{\pi^+}(a_0 - a_2) = 0.249 \pm 0.004_{(stat)} \pm 0.004_{(ext)} \pm 0.013_{(th.)}$$

$$\chi^2/n.d.f = 2120.6/2037 \sim 1.04$$

This is the main result of this analysis and it agrees with the NA48/2 value at the level of 1σ .

The same correlation between parameters has been obtained with a totally different set-up by KTeV experiment collaboration at Fermilab that published, in June 2008, a similar study of the $K_L^0 \rightarrow 3\pi^0$ decay Dalitz plot density, based on a sample of about 68 million events and described by the same rescattering model (Ref. [49]). The KTeV results are:

- Fit for h with fixed $a_0 - a_2$:

$$h_{\text{KTeV}} = (+0.59 \pm 0.20_{\text{stat.}} \pm 0.48_{\text{syst.}} \pm 1.06_{\text{ext.}}) \cdot 10^{-3}$$

$$\chi^2/n.d.f = 2805.3/2765 \sim 1.01$$

- Fit for h and $a_0 - a_2$:

$$h_{\text{KTeV}} = (-2.09 \pm 0.62_{\text{stat.}} \pm 0.72_{\text{syst.}} \pm 0.28_{\text{ext.}}) \cdot 10^{-3}$$

$$m_{\pi^+}(a_0 - a_2)_{\text{KTeV}} = 0.215 \pm 0.014_{\text{stat.}} \pm 0.025_{\text{syst.}} \pm 0.006_{\text{ext.}}$$

$$\chi^2/n.d.f = 2805.3/2765 \sim 1.01$$

The correlation between parameters obtained by KTeV is even larger (about 94%). However, both analyses give for h a result consistent with zero that disagrees with the pre-rescattering PDG value. In Fig.10.1-10.2 we list all available experimental results for $a_0 - a_2$ and h respectively, including the values obtained by KTeV and the present analysis.

From this analysis we demonstrate that the cusp effect in the $K_L^0 \rightarrow 3\pi^0$ decay allows to extract the scattering length $a_0 - a_2$ with a good accuracy although the size of the cusp in the K_L^0 system is suppressed by a factor 13 with respect to the $K^+ \rightarrow \pi^+\pi^0\pi^0$ decay. This work is still preliminary but it encourages a subsequent detailed study of the systematics that we have not investigate in this thesis.

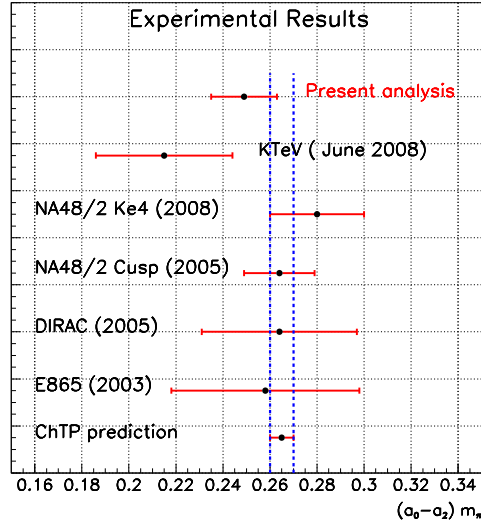


Figure 10.1: *Experimental results for $a_0 - a_2$. We see that our value is consistent with the ChPT prediction and the other experimental results. In particular, the difference between our result and KTeV value should be noted*

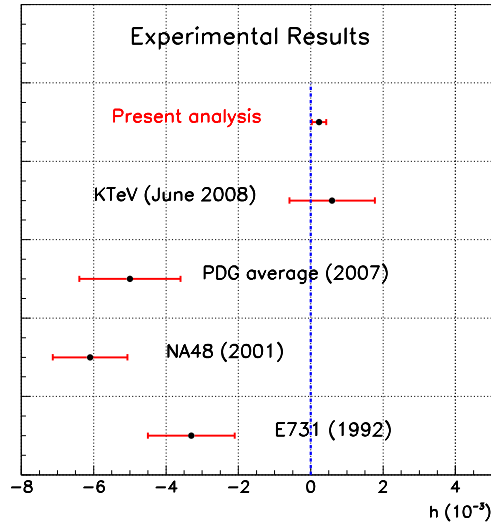


Figure 10.2: *Experimental results for h . Our result refers to the value of h obtained by fixing $a_0 - a_2$ (the error bar in this case is only due to the statistical error)*

Appendix A

Isotopic relations for $K \rightarrow 3\pi$ decays and $\Delta I = 1/2$ rule

We denote a pion as π_i , $i = 1, 2, 3$ and the respective final state as $|\pi_i, m_i\rangle$, where $m_i = \pm, 0$ indicates the isospin z -component. It's necessary to classify the possible 3π states by their total isotopic spin; in general, three pions may form a state with $I = 0, 1, 2$ and 3 :

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \underbrace{\mathbf{1}}_{I=0} \oplus \underbrace{\mathbf{3} \oplus \mathbf{3} \oplus \mathbf{3}}_{I=1} \oplus \underbrace{\mathbf{5} \oplus \mathbf{5}}_{I=2} \oplus \underbrace{\mathbf{7}}_{I=3}$$

The states of total isospin $|I, m\rangle$ are obtained by making a repeated use of the Clebsch-Gordan coefficients:

$$|I_{12}, m_{12}\rangle = \sum_{m_1, m_2} C_{m_{12}, m_1, m_2}^{I_{12}, I_1, I_2} \times |\pi_1, m_1\rangle |\pi_2, m_2\rangle \quad (\text{A.1})$$

$$|I, m\rangle = \sum_{I_{12}} \sum_{m_3, m_{12}} C_{m, m_3, m_{12}}^{I, I_3, I_{12}} \times |\pi_3, m_3\rangle |I_{12}, m_{12}\rangle \quad (\text{A.2})$$

where $m = m_1 + m_2 + m_3$. The states with $I = 0$ and $I = 3$ are completely anti-symmetric and symmetric, respectively. Two states with $I = 2$ have mixed symmetry. One of the three states with $I = 1$ is symmetric while the remaining two have mixed symmetry. In order to exploit the Bose symmetry, the space wave functions are also needed. As pointed out by Okubo *et al.* (Ref. [2]) and Gell-Mann and Rosenfeld (Ref. [50]), we can make the assumption that the space wave function is approximately symmetric (the pions are emitted in S-wave due to the decay low-energy release $Q = M_K - 3m_\pi$) also justified by the experimental results ($K \rightarrow 3\pi$ Dalitz plot is nearly symmetric).

A more accurate treatment was given by Barton *et al.* (Ref. [4]) where the space dependence on s_i variables was assumed at most linear. By assuming the space wave functions to be completely symmetric, Bose principle rules out the states with $I = 0$ and $I = 1, 2$ (with mixed symmetry) but nothing can be said for the remaining states $I = 1, 3$.

In general, the interaction hamiltonian that give rise to $K \rightarrow 3\pi$ will be an admixture of $\Delta I = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$ and can be written as:

$$H = H_{1/2} + H_{3/2} + H_{5/2} + H_{7/2}$$

where $H_{n/2}$ behaves like a $I = n/2$ quantity under isotopic rotation. However, because of the state is a mixing of $I = 1, 3$, the hamiltonian contains no admixtures of $\Delta I = \frac{3}{2}, \frac{5}{2}$ but only $\Delta I = \frac{1}{2}, \frac{7}{2}$ terms.

If the isotopic selection rule $\Delta I = \frac{1}{2}$ holds and we can neglect any dependence on s_i , from Eq. (A.2), we find:

$$\frac{|\mathcal{M}_{++-}|^2}{|\mathcal{M}_{+00}|^2} = 4 \quad (\text{A.3})$$

$$\frac{|\mathcal{M}_{+-0}|^2}{|\mathcal{M}_{000}|^2} = \frac{2}{3} \quad (\text{A.4})$$

This is the $\Delta I = \frac{1}{2}$ rule prediction for the rate ratios. We obtain the isospin amplitudes $A^+/A^0 = 2$ and $A_L^+/A_L^0 = 1/3$ by taking the square root (these amplitudes are defined to contain also the Bose factor as shown in Chapter 9). The observed branching ratio taken from PDG are comparable with these predictions ¹:

$$\frac{\Gamma^{PDG}(K^+ \rightarrow \pi^+\pi^+\pi^-)}{\Gamma^{PDG}(K^+ \rightarrow \pi^+\pi^0\pi^0)} = \frac{(5.590 \pm 0.031)\%}{(1.757 \pm 0.024)\%} = 3.182 \pm 0.047$$

$$\frac{\Gamma^{PDG}(K_L^0 \rightarrow \pi^+\pi^-\pi^0)}{\Gamma^{PDG}(K_L^0 \rightarrow 3\pi^0)} = \frac{(12.56 \pm 0.05)\%}{(19.56 \pm 0.14)\%} = 0.642 \pm 0.005$$

On the contrary, if H is assumed to be pure $\Delta I = \frac{3}{2}$ we would predict (Ref. [4]) $A^+/A^0 = 8$ in explicit contradiction with the experiments. Furthermore, if a linear kinematical dependence on s_i variables is considered, some relations on the size of the (linear) slopes can be derived ((Ref. [3], [4]). For example, the following relations are predicted (see Chapter 3 for notation):

¹These ratio should be corrected for the space-space volumes due to differences in mass on kaons and pions and for kinematical factors that we have discarded in first approximation.

$$2g^+ = -g^0$$

$$h = 0$$

compared with the PDG values:

$$\begin{aligned} g^0 &= 0.626 \pm 0.007 \\ g^+ &= -0.2154 \pm 0.0035 \\ h &= (-5.0 \pm 1.4) \cdot 10^{-3} \end{aligned}$$

The rate relations (Eq. (A.3), A.4) for $K \rightarrow 3\pi$ decays can be derived more easily by using the isotopic vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ to represent the isospin amplitudes of the first, second and third pions, respectively (Ref. [5], [51]). Charge amplitudes are defined as:

$$\mathbf{a} = (a_+, a_0, a_-)$$

We can always decompose a generic amplitude as sum of a products of the form:

$$\mathcal{M} = \sum \mathcal{M}_I \mathcal{M}_P \mathcal{M}_F$$

where $\mathcal{M}_I$ carries the isotopic spin dependence, $\mathcal{M}_{JP}$ carries the spin J and the parity P and $\mathcal{M}_F$ the remaining momentum-energy dependence (form factor). To construct $\mathcal{M}_I$, we combine the three isospin vectors to form irreducible quantities (states) under a isotopic rotation; for example, for $I = 0$ and $I = 1$ we have:

- Case $I = 0$

$$\mathcal{M}_I = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \epsilon_{ijk} a_i b_j c_k$$

- Case $I = 1$

$$\mathcal{M}_I = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$$

In case $I = 0$, the Bose statistic dictates the form factor to be a completely anti-symmetric function of the s_i variables, while in case $I = 1$, a function symmetric in a couple of pions. Because we are interested in the $\Delta I = \frac{1}{2}$ transition ($I=1$), we obtain:

$$\mathcal{M} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) A + \mathbf{b}(\mathbf{c} \cdot \mathbf{a}) B + \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) C \quad (\text{A.5})$$

that is symmetric in the three pions; A , B , C are symmetric in the pions (2,3), (1,3) and (1,2), respectively. Then we have:

$$\frac{|\mathcal{M}_{++-}|^2}{|\mathcal{M}_{+00}|^2} = \frac{|A+B|^2}{|C|^2}$$

$$\frac{|\mathcal{M}_{++-}|^2}{|\mathcal{M}_{+00}|^2} = \frac{2(|A|^2 + |B|^2 + |C|^2)}{|A+B+C|^2}$$

Because of the Q value is low, the functions A , B , C are substantially constant and equal to one another so that we find again the rate relations of Eq. (A.3), A.4.

Appendix B

Unitarity and analyticity of the S-matrix

The unitarity condition for the S -matrix reads:

$$S S^\dagger = S^\dagger S = 1 \quad (\text{B.1})$$

The T -matrix is defined as:

$$S - 1 \equiv +i T$$

so that Eq. (B.1) becomes:

$$-i(T - T^\dagger) = T^\dagger T$$

Inserting a complete set of states $|n\rangle$, we obtain:

$$\mathcal{M}(i \rightarrow f) - \mathcal{M}^*(f \rightarrow i) = \sum_n i \mathcal{M}^*(f \rightarrow n) \mathcal{M}(i \rightarrow n) \quad (\text{B.2})$$

where $\mathcal{M}(i \rightarrow f) = \langle f|T|i\rangle$ and $\sum$ denotes a sum and an integral over all intermediate states that are allowed by conservation of the total energy, momentum and other possible selection rules; for $i = f$ we obtain the *optical theorem*. Above the energy threshold for inelastic scattering a new term must be added to right-hand side of Eq. (B.2) so as to include the extra intermediate states. This implies a change in the left-side and suggests that the elastic scattering matrix-element has a singularity at each energy corresponding to a threshold for a new allowed physical process (“normal-threshold singularities”). For example, for two particles $\rightarrow$ two particles scattering, the thresholds are branch-points of the amplitude in the complex energy-squared plane $s = E_{tot}^2$ (E_{tot} = centre of mass energy) (Ref. [52]); then, in order to make the amplitude single-valued on a Riemann surface, we must draw cuts attached to these branch points (Fig. B.1).

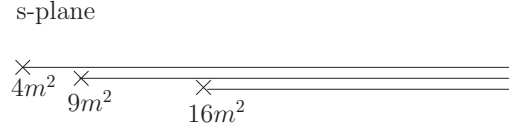


Figure B.1: *Branch cuts in the s -plane for the scattering amplitude involving identical spinless particles. The cuts arise from normal thresholds $s = 4m^2, 9m^2, 16m^2, \dots$ that correspond to the energies at which production of extra particles is possible.*

The discontinuity at a given threshold $s_{\text{threshold}}$ is the difference of the amplitude crossing the cut: ¹

$$\text{Disc}(\mathcal{M})^{\text{th.}} = \mathcal{M}(s + i\epsilon) - \mathcal{M}(s - i\epsilon)$$

where ϵ is an infinitesimal quantity and $s > s_{\text{threshold}}$ is real and lies on the cut. It can be shown that $\mathcal{M}(s - i\epsilon) = \mathcal{M}^*(f \rightarrow i)$, $\forall s$ (“hermitian analyticity”) so that the unitarity condition Eq. (B.2) evaluates discontinuities (Ref. [52]):

$$\text{Disc}(\mathcal{M}(i \rightarrow f)) = \sum_n i \mathcal{M}^*(f \rightarrow n) \mathcal{M}(i \rightarrow n) \quad (\text{B.3})$$

In general, the amplitude $\mathcal{M}$ is given as a perturbative expansion in terms of Feynman diagrams. According to Eq. (B.3), the discontinuity of a diagram associated with a given threshold can be evaluated by stripping off all the internal virtual particles (propagators) representing the new allowed intermediate state $|n\rangle$ and putting them on shell: the diagram will result disconnected in two pieces as requested by Eq. (B.3) (Fig. B.2). Finally, the total discontinuity $\text{Disc}(\mathcal{M})^{\text{th.}}$ is obtained by summing all the diagrams cut in this way. These are the Cutkosky rules (Ref. [30]). ²

¹In general, the amplitude is function of other variables (i.e. s, t, u); we are considering $\mathcal{M}$ as function of s keeping t fixed.

²So far, we have discussed only the singularities that can be deduced from unitarity (normal-threshold singularities); however, in a diagram other singularities, associated with a given set of propagators, may occur.

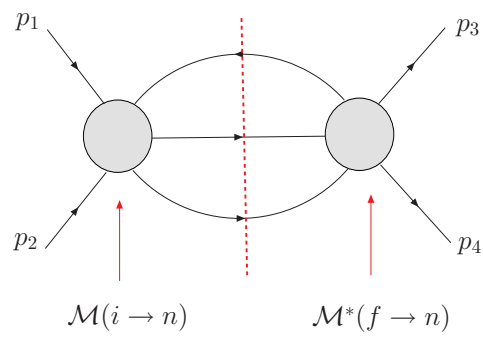


Figure B.2: *Cutkosky rules*

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